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A GENERAL METHOD FOR ESTIMATING DISTRIBUTION PARAMETERS

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Switzerland

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FOREWORD

This report was prepared by Prof. Dr. Waloddi Weibull, Lausanne, Switzerland, under USAF Contract No. AF 61(052)-943. The contract was initiated under Project No. 7351, "Metallic Materials," Task No. 735106, "Behavior of Metals." The contract was administered by the European Office, Office of Aerospace Research. The work was monitored by the Air Force Materials Laboratory, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, under the direction of Mr. W. J. Trapp.

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This technical report has been reviewed and is approved.

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ABSTRACT

The method presented is applicable to complete, censored, or truncated samples and to grouped data drawn from any population having a continuous distribution function, simple or composed, involving an arbitrary number of unknown parameters. The estimates are consistent and asymptotically efficient (in some cases for any sample size) and easily determined by use of a versatile computer program. The efficiency can be stated for any individual case, even when only a part of the sample is used for the estimation. Two criteria of goodness-of-fit, which complete each other, makes it possible to decide whether the fit attained is acceptable or not.

Two of its many applications may be mentioned: the evaluation of data from bending and torsional tests on brittle materials, a problem up-to-now not quite satisfactorily solved due to the complicated distribution functions arising; and the analysis of bimodal fatigue-life distributions.

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1. Introduction

After having successfully applied to a number of large samples from mixed distribution a new method based on a minimum-seeking computer program and having found that the computing times, in most cases, were surprisingly short even with several unknown parameters, it was decided to examine the applicability of this method also to simple distributions and to the more complicated ones pertaining to statistical strengths of brittle materials subjected to bending or torsion.

As will be demonstrated below this method has a very general applicability, and the estimates are consistent and at least asymptotically efficient.

The numerical evaluation of one, two or three unknown parameters of a simple distribution function is, by means of a digital computer, a matter of some twenty, thirty seconds only.

2. General formulas

Let the Cdf of the random variable X be

$$P(X \leq x) = F(x,a,b,c) \tag{1}$$

where a,b,c are the unknown parameters.

The method consists in the determination, by use of a minimum-seeking computer program, of that set $\hat{a},\hat{b},\hat{c}$ which satisfies the condition

$$M = \Sigma[w_i(F(x_i,\hat{a},\hat{b},\hat{c}) - p_i)^2] = \text{minimum}$$

where x_i are the observations, and the weights w_i and the constants p_i are determined in such a way that the estimates are consistent and have minimum variance.

The importance of this method is due to the facts that w_i can be determined in such a way that the estimates will be asymptotically efficient and that the estimation can be easily performed by use of a computer program.

The asymptotic properties of this estimation method will now be examined.

Let the differences Δa, Δb, Δc between the exact and the estimated values of the parameters be defined by

$$\hat{a} = a - \Delta a \quad ; \quad \hat{b} = b - \Delta b \quad ; \quad \hat{c} = c - \Delta c \tag{3}$$

Then, if the squares of these differences can be neglected, as being the case for large enough samples, say $N \geq 50$, the sum (2) can be written

$$M = \Sigma[w_i(F(x_i,a,b,c) - f_{ai}\,\Delta a - f_{bi}\,\Delta b - f_{ci}\,\Delta c - p_i)^2]$$

where (4)

$$f_a = \partial F/\partial a \quad ; \quad f_b = \partial F/\partial b \quad ; \quad f_c = \partial F/\partial c$$

It is a well known fact that, if x_i are the elements of a sample drawn from a population having the Cdf (1), then the random variable

$$r_i = F(x_i,a,b,c) \tag{5}$$

is distributed exactly as the i:th order statistic of a sample drawn from a population which is rectangularly distributed.

Hence

$$\left.\begin{aligned} E[F(x_i,a,b,c)] &= i/(N+1) \\ \mathrm{Cov}[F(x_i,a,b,c),F(x_j,a,b,c)] &= i(N+1\underset{i \leq j}{} - j)/(N+1)^2(N+2) \end{aligned}\right\} \tag{6}$$

including the variance for $i = j$.

If G denotes the inverse of the function F, then

$$x_i = G(r_i) \tag{7}$$

For increasing sample sizes

$$r_i \longrightarrow i/(N+1) \quad \text{and thus} \quad x_i \longrightarrow G[i/(N+1),a,b,c]$$

Consequently, for large samples sizes we may substitute for the coefficients in (4), actually being $f_a(x_i,a,b,c)$ etc., and thus depending on the observations x_i, the constants

$$f_{ai} = f_a[G(i/(N+1),a,b,c)] \quad \text{etc.} \tag{8}$$

which are, for a given distribution, uniquely determined by i and N.

For a given set w_i the sum (4) is minimized by the set Δa, Δb, Δc which satisfies the conditions

$$\left.\begin{aligned}
&\Sigma[w_i f_{ai}(r_i - p_i - f_{ai}\,\Delta a - f_{bi}\,\Delta b - f_{ci}\,\Delta c)] = 0\\
&\Sigma[w_i f_{bi}(r_i - p_i - f_{ai}\,\Delta a - f_{bi}\,\Delta b - f_{ci}\,\Delta c)] = 0\\
&\Sigma[w_i f_{ci}(r_i - p_i - f_{ai}\,\Delta a - f_{bi}\,\Delta b - f_{ci}\,\Delta c)] = 0
\end{aligned}\right\} \tag{9}$$

With the notations

$$\left.\begin{aligned}
&\Sigma[w_i f_{ai}^2] = k_a \qquad &&\Sigma[w_i f_{bi}^2] = k_b \qquad &&\Sigma[w_i f_{ci}^2] = k_c\\
&\Sigma[w_i f_{ai} f_{bi}] = k_{ab} \qquad &&\Sigma[w_i f_{ai} f_{ci}] = k_{ac} \qquad &&\Sigma[w_i f_{bi} f_{ci}] = k_{bc}
\end{aligned}\right\} \tag{10}$$

the system (9) becomes

$$\left.\begin{aligned}
k_a\,\Delta a + k_{ab}\,\Delta b + k_{ac}\,\Delta c &= \Sigma[w_i f_{ai}(r_i - p_i)]\\
k_{ba}\,\Delta a + k_b\,\Delta b + k_{bc}\,\Delta c &= \Sigma[w_i f_{bi}(r_i - p_i)]\\
k_{ca}\,\Delta a + k_{cb}\,\Delta b + k_c\,\Delta c &= \Sigma[w_i f_{ci}(r_i - p_i)]
\end{aligned}\right\} \tag{11}$$

From the determinant

$$A = \begin{vmatrix} k_a & k_{ab} & k_{ac}\\ k_{ba} & k_b & k_{bc}\\ k_{ca} & k_{cb} & k_c \end{vmatrix} \tag{12}$$

and its cofactors A_{ij} $(i,j=a,b,c)$ we have

$$\Delta a = \Sigma[(A_{aa}f_{ai} + A_{ab}f_{bi} + A_{ac}f_{ci})w_i(r_i - p_i)]/A \quad \text{etc.} \tag{13}$$

With the notations

$$\left.\begin{aligned} h_{ai} &= (A_{aa}f_{ai} + A_{ab}f_{bi} + A_{ac}f_{ci})w_i/A \\ h_{bi} &= (A_{ba}f_{ai} + A_{bb}f_{bi} + A_{bc}f_{ci})w_i/A \\ h_{ci} &= (A_{ca}f_{ai} + A_{cb}f_{bi} + A_{cc}f_{ci})w_i/A \end{aligned}\right\} \tag{14}$$

we have

$$\Delta a = \Sigma[h_{ai}(r_i - p_i)] \quad ; \quad \Delta b = \Sigma[h_{bi}(r_i - p_i)] \quad ; \quad \Delta c = \Sigma[h_{ci}(r_i - p_i)] \tag{15}$$

These formulas are useful for appraising the present method, but they are not intended for numerical computations, which will be performed by use of a minimum-seeking computer program directly applied to the sum (2).

From (15) the expected values

$$E(\Delta a) = \Sigma[h_{ai}(i/(N+1) - p_i)] \quad \text{etc.,}$$

evidently are equal to zero on the condition that

$$p_i = i/(N+1) \tag{16}$$

and thus by (3)

$$E(\hat{a}) = a \qquad E(\hat{b}) = b \qquad E(\hat{c}) = c \tag{17}$$

that is, if (16) is introduced into (2), then the estimates will be consistent.

Considering that $Var(\hat{a}) = Var(\Delta a)$ etc., it follows that

$$\left.\begin{aligned}(N+1)^2(N+2)\mathrm{Var}(\hat{a}) &= \Sigma[h_{ai}\cdot h_{aj}\cdot i(N+1-j)]\\ (N+1)^2(N+2)\mathrm{Var}(\hat{b}) &= \Sigma[h_{bi}\cdot h_{bj}\cdot i(N+1-j)]\\ (N+1)^2(N+2)\mathrm{Var}(\hat{c}) &= \Sigma[h_{ci}\cdot h_{aj}\cdot i(N+1-j)]\end{aligned}\right\} \tag{18}$$

The sums are to be interpreted as

$$\Sigma = \sum_{i=j} + 2.\sum_{i<j} \tag{19}$$

It should be observed that the variances computed by (18) are lower bounds which, for small samples, are unattainable due to the fact that the scatter in the coefficients f_{ai}, f_{bi}, etc. has been neglected. With increasing sample size the formulas tend to the correct values.

3. Unweighted observations

For any given set w_i, in particular for unweighted observations, that is, $w_i = 1$, including also the case when some of the observations have been omitted $(w_i = 0)$, the variances of the estimates are easily computed.

The formulas will be used, as illustrated in Chapter 7, for stating the loss in efficiency of the estimates, if the proper weighting of the observations is neglected.

It should be observed that in the particular case when the number of unknown parameters is equal to the number of observations used for estimation, each term of the sum (2) is, independently of w_i, equal to zero. Thus we may, in this case, put $w_i = 1$.

Since this case frequently occurs as being convenient for obtaining suitable starting values of the estimates for the minimum-seeking computer program, it is of interest to determine which of the order numbers that will provide the least variances.

4. Efficient weighting of the observations

For an arbitrary number of unknown parameters $a, b, c, \ldots,$

the first equation of (11) takes the form

$$k_a \triangle a + k_{ab} \triangle b + k_{ac} \triangle c + . . . = \Sigma[w_i f_{ai}(r_i - p_i)] \tag{20}$$

In order to minimize $Var(\hat{a})$ by a proper weighting let us tentatively examine the consequences of the conditions

$$\Sigma[w_i f_{ai}^2] = k_a = 1 \quad \Sigma[w_i f_{ai} f_{bi}] = k_{ab} = 0 \quad \Sigma[w_i f_{ai} f_{ci}] = k_{ac} = 0 \tag{21}$$

Then equ.(20) takes the simple form

$$\triangle a = \Sigma[w_i f_{ai}(r_i - p_i)] \tag{22}$$

and

$$(N+1)^2(N+2)Var(\hat{a}) = \Sigma[w_i f_{ai} . w_j f_{aj} . i(N+1-j)] \tag{23}$$

where the summation is defined by (19).

Denoting

$$w_i' = w_i . f_{ai} \tag{24}$$

it is required that

$$(N+1)^2(N+2)Var(\hat{a}) = \Sigma[w_i' . w_j' . i(N+1-j)] = \text{minimum} \tag{25}$$

with the side conditions

$$\Sigma[w_i' f_{ai}] = 1 \quad \Sigma[w_i' f_{bi}] = 0 \quad \Sigma[w_i' f_{ci}] = 0 \tag{26}$$

Hence, derivating (25) with regard to w_i' and equating to zero

$$\left.\begin{array}{l} w_1' . 1 . N \quad + w_2' . 1(N-1) + . . . + w_N' . 1 . 1 = \lambda_a f_{a1} + \lambda_b f_{b1} + \lambda_c f_{c1} + . \\ w_1' . 1(N-1) + w_2' . 2(N-1) + . . . + w_N' . 2 . 1 = \lambda_a f_{a2} + \lambda_b f_{b2} + \lambda_c f_{c2} + . . \\ - - - - - - - - - - - - - - - - - - - \\ w_1' . 1 . 1 \quad + w_2' . 2 . 1 \quad + . . . + w_N' . N . 1 = \lambda_a f_{aN} + \lambda_b f_{bN} + \lambda_c f_{cN} + . . \end{array}\right\} \tag{27}$$

Multiplying each of the equs (27) by w_i', adding and considering (26) it follows by (23) that

$$(N+1)^2(N+2)\mathrm{Var}(\hat{a}) = \lambda_a \tag{28}$$

The matrix B, pertinent to the system (27)

$$B = \begin{bmatrix} 1 \cdot N & 1(N-1) & 1(N-2) & \dots & 1 \cdot 1 \\ 1(N-1) & 2(N-1) & 2(N-2) & \dots & 2 \cdot 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 \cdot 1 & 2 \cdot 1 & 3 \cdot 1 & \dots & N \cdot 1 \end{bmatrix} \tag{29}$$

is square and centrosymmetric with the principal diagonal elements equal to $i(N-1-i)$, as illustrated by the matrix for $N = 9$, being of the order 9x9.

$$\begin{bmatrix} \boxed{9} & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 8 & \boxed{16} & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 7 & 14 & \boxed{21} & 18 & 15 & 12 & 9 & 6 & 3 \\ 6 & 12 & 18 & \boxed{24} & 20 & 16 & 12 & 8 & 4 \\ 5 & 10 & 15 & 20 & \boxed{25} & 20 & 15 & 10 & 5 \\ 4 & 8 & 12 & 16 & 20 & \boxed{24} & 18 & 12 & 6 \\ 3 & 6 & 9 & 12 & 15 & 18 & \boxed{21} & 14 & 7 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & \boxed{16} & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \boxed{9} \end{bmatrix}$$

This matrix can by use of elementary operations be transformed into an equivalent diagonal matrix, as will be demonstrated for the more general case that some of the rows and corresponding columns are suppressed.

The following notations will be used:

Row number v	order number i_v	Diff. d_v	Row R_v	Matrix		
1	i_1	$d_1 = i_1$	R_1	$\boxed{i_1(N+1-i_1)}$	$i_1(N+1-i_2)$. .	. $i_1(N+1-i_V)$
2	i_2	$d_2 = i_2 - i_1$	R_2	$i_1(N+1-i_2)$	$\boxed{i_2(N+1-i_2)}$ . .	. $i_2(N+1-i_V)$
. . .	. . .	. . .	. . .	. . .	. . .	. . .
V	i_V	$d_V = i_V - i_{V-1}$	R_V	$i_1(N+1-i_V)$	$i_2(N+1-i_V)$. .	. $\boxed{i_V(N+1-i_V)}$
–	–	$d_{V+1} = N+1-i_V$	–	–	–	–

Since $i_1 = d_1$ $i_2 = d_1 + d_2$ $i_3 = d_1 + d_2 + d_3$ etc., the matrix is uniquely specified by the set d_v. For the complete matrix, where $i_v = v$ and $i_V = N$, that is, the row numbers and the order numbers are identical, we have

$$d_v = d_i = 1 \tag{30}$$

Let us now take three <u>consecutive</u> rows with the order numbers b,c,d and two more rows with the arbitrary order numbers x,y satisfying $x < b < c < d < y$. This part of the matrix with the diagonal elements framed is

$$R_{v-1} = [x(N+1-b)\ldots \boxed{b(N+1-b)}\ b(N+1-c)\quad b(N+1-d)\ldots\ b(N+1-y)]$$

$$R_v \ = [x(N+1-c)\ldots\ b(N+1-c)\ \boxed{c(N+1-c)}\ c(N+1-d)\ldots\ c(N+1-y)]$$

$$R_{v+1} = [x(N+1-d)\ldots\ b(N+1-d)\quad c(N+1-d)\ \boxed{d(N+1-d)}\ldots\ d(N+1-y)]$$

If now to the row R_v is applied an operation denoted by $Op(R_v)$ and defined by

$$Op(R_v) = (d-b)R_v - (d-c)R_{v-1} - (c-b)R_{v+1}$$

that is, the elements of R_{v-1} are multiplied by $d_{v+1} = d - c$, those of R_v by $(d_v + d_{v+1}) = d - b$ and those of R_{V+1} by $d_v = c - b$, then it is easy to verify that the diagonal element of R_v will be $d_v \cdot d_{v+1}(N+1)$ and that all other elements of this row will disappear, which may be symbolized by

$$Op(R_v) = (d_v + d_{v+1})R_v - d_{v+1}R_{v-1} - d_vR_{v+1} = d_vd_{v+1}(N+1) \tag{31}$$

For a complete matrix, where $d_v = d_i = 1$, the operation applied to any row R_i will result in

$$Op(R_i) = 2R_i - R_{i-1} - R_{i+1} = (N+1) \tag{32}$$

By use of operation (31) the system (27) reduces to

$$w'_v d_v d_{v+1}(N+1) = \lambda_a Op(f_{av}) + \lambda_b Op(f_{bv}) + \lambda_c Op(f_{cv}) + \cdots \tag{33}$$

By (24) the weights w_i then become

$$f_a \cdot w_v = [\lambda_a Op(f_{av}) + \lambda_b Op(f_{bv}) + \lambda_c Op(f_{cv}) + \ldots]/(N+1)d_v d_{v+1} \tag{34}$$

and for the complete matrix

$$w_i = [\lambda_a Op(f_{ai}) + \lambda_b Op(f_{bi}) + \lambda_c Op(f_{ci}) + \ldots]/(N+1) \cdot f_{ai} \tag{35}$$

With the notations

$$AA = \Sigma[f_{av} Op(f_{av})/d_v d_{v+1}] \qquad AB = \Sigma[f_{av} Op(f_{bv})/d_v d_{v+1}] \tag{36}$$

the values of $\lambda_a, \lambda_b, \lambda_c$ will be determined as the roots of the system of equations obtained by introducing (33) into (26) and being

$$\left.\begin{array}{l} AA \cdot \lambda_a + AB \cdot \lambda_b + AC \cdot \lambda_c + \ldots = N+1 \\ BA \cdot \lambda_a + BB \cdot \lambda_b + BC \cdot \lambda_c + \ldots = 0 \\ CA \cdot \lambda_a + CB \cdot \lambda_b + CC \cdot \lambda_c + \ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \tag{37}$$

from which it follows that

$$\lambda_a = (N+1)\begin{vmatrix} BB & BC & \cdot & \cdot \\ CB & CC & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} : \begin{vmatrix} AA & AB & AC & \cdot & \cdot \\ BA & BB & BC & \cdot & \cdot \\ CA & CB & CC & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}. \tag{38}$$

Using the definition (32) of the operations it is readily seen that $AB = BA$ $AC = CA$ etc.

Introducing (39) into (28)

$$(N+1)(N+2)\mathrm{Var}(a) = \begin{vmatrix} BB & BC & \cdot & \cdot \\ CB & CC & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} : \begin{vmatrix} AA & AB & AC & \cdot & \cdot \\ BA & BB & BC & \cdot & \cdot \\ CA & CB & CC & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \qquad (39)$$

Further from (37) the relative values of λ_a, λ_b, λ_c, which are required for computing w_v by (34)

$$\lambda_a : \lambda_b : \lambda_c : \ldots = \begin{vmatrix} BB & BC & \cdot \\ CB & CC & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} : -\begin{vmatrix} BA & BC & \cdot \\ CA & CC & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} : \begin{vmatrix} BA & BB & \cdot \\ CA & CB & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} . \qquad (40)$$

It should be observed that the weighting of the observations, resulting from introducing (40) into (35) will minimize $\mathrm{Var}(\hat{a})$ but not $\mathrm{Var}(\hat{b})$ and $\mathrm{Var}(\hat{c})$ which, however can be minimized by other analogous sets of w_i. Thus, in order to attain maximum efficiency, a separate set of weights is required for each unknown parameter. This problem will be illustrated by some numerical examples in the sequel.

In order to state the efficiency of the estimates the relation between the symbols AA, AB, etc., and the Fisher concept "amount of information provided by a random sample" will be examined.

5. Efficiency of the estimates

For the complete matrix the symbol AA is by (36) defined by

$$AA = \sum_{i=1}^{N}\left[f_{ai}\,\mathrm{Op}(f_{ai})\right] \qquad f_{ao} = f_{a,N+1} = 0 \qquad (41)$$

From the definition $\mathrm{Op}(f_{ai})$, equ.(41) is easily transformed into

$$AA = f_{a1}^2 + \sum_{i=1}^{N-1}(f_{a,i+1} - f_{ai})^2 + f_{aN}^2 \qquad (42)$$

and in the same way

$$AB = f_{a1} \cdot f_{b1} + \sum_{i=1}^{N-1}(f_{a,i+1} - f_{ai})(f_{b,i+1} - f_{bi}) + f_{aN} \cdot f_{bN} \qquad (43)$$

The sum in (42) may be written

$$\Sigma(f_{a,i+1} - f_{ai})^2 = \Sigma\left(\frac{[f_a(x_{i+1}) - f_a(x_i)]\,\Delta x_i}{\Delta x_i} \right)^2$$

but since

$$r = F(x,a,b,c) \quad \text{and} \quad dr = f(x,a,b,c)$$

thus for large enough samples

$$\Delta x_i = \Delta r_i / f = 1/(N+1)f$$

and

$$\sum_{i=1}^{N-1}(f_{a,i+1} - f_{ai})^2 = \frac{1}{N+1}\,\Sigma\left(\frac{f_a(x_{i+1}) - f_a(x_i)}{\Delta x_i} \right)^2 \frac{\Delta x_i}{f}$$

On the condition that the function f_a is continuous and bounded

$$\Sigma(f_{a,i+1} - f_{ai})^2 \longrightarrow \frac{1}{N+1} \int_0^\infty \left(\frac{\partial f_a}{\partial x}\right)^2 \frac{dx}{f} \quad \text{as} \quad N \longrightarrow \infty \tag{44}$$

Since by definition $f_a = \partial F/\partial a$ we have

$$\frac{\partial f_a}{\partial x} = \frac{\partial^2 F}{\partial x\,\partial a} = \frac{\partial^2 F}{\partial a\,\partial x} = \frac{\partial f}{\partial a} \tag{45}$$

and then for large samples

$$\Sigma(f_{a,i+1} - f_{ai})^2 = \frac{1}{N} \int_0^\infty \left(\frac{f}{a}\right)^2 \frac{dx}{f} \tag{46}$$

Observing that the Fisher amount of information is defined by

$$I = N \int_0^\infty \left(\frac{\partial f}{\partial a}\right)^2 \frac{dx}{f} \tag{47}$$

it follows in the regular case, which includes not only that the integral (47) exists but also that f_{a1} and f_{aN} tend to zero as $N \to \infty$, that

$$AA = I/N^2 \tag{48}$$

and, for one unknown parameter that

$$Var(\hat{a}) = 1/I \tag{49}$$

which proves that the estimate is asymptotically efficient. This result can be immediately extended to any number of unknown parameters. Thus, it can be concluded that no conditions differing from (21) can provide more efficient estimates.

The non-regular case will be examined in Chapter 7 in connection with the applications to the Weibull distribution.

6. Two criteria for goodness-of-fit

Two statistics, presenting themselves so-to-say as by-products of the estimation procedure, can with advantage be used as criteria of goodness-of-fit. Both of them employ a minimum of assumptions, merely that the sample is randomly drawn from the population in question.

The first one is the minimum-sum M_{min} actually attained for an individual sample. The characteristics of this statistic can be determined and, if the actual value of this sum differs too much from its expected value, the fit has to be rejected. The difference that can be tolerated will be set in relation to the variance of M_{min} or, when its distribution is known, in relation to some accepted tolerance interval.

The second statistic is the number-of-runs U. Observing that, in order to determine the sum M_{min}, each deviation $(F(x_i,\hat{a},\hat{b},\hat{c}) - p_i)$ is computed, squared and weighted, the signs of these deviations are directly obtained and also the sequences of terms having the same sign. Any such sequence is called a run. Now, if the deviations are purely random, the distribution and the expected value of the number-of runs U is known for any sample size, and this number may be used as another criterion.

These two criteria make a good complement of each other because the first one reacts for large, unsystematic deviations, while the second one reacts for systematic deviations how small they ever may be. If M_{min} is too large and U too small, either the randomness of the sampling or lack of control of the testing procedure or the assumed distribution will be suspected. If, on the other hand, M_{min} is very small or U very large, then the randomness will be suspected due to, for instance, some smoothing of the data or the like.

Pertinent formulas will be presented in a subsequent report.

7. Application to the Weibull distribution

7.1 General formulas

From the cumulative distribution function

$$F(x) = 1 - e^{-[(x-\mu)/\beta]^{1/\alpha}} \tag{50}$$

the coefficients $f_{\alpha i}$, $f_{\beta i}$, $f_{\mu i}$ defined by (8) are obtained by derivating $F(x)$ with respect to α, β, μ and inserting the values x_i which satisfy the condition

$$e^{-[(x_i-\mu)/\beta]^{1/\alpha}} = 1 - p_i \tag{51}$$

that is,

$$(x_i - \mu)/\beta = [-\ln(1-p_i)]^{\alpha} \tag{52}$$

where according to (16) $p_i = i/(N+1)$.

In this way we have

$$\alpha \cdot f_{\alpha i} = (1-p_i)\ln(1-p_i)\ln[-\ln(1-p_i)] \tag{53}$$

or substituting $m = 1/\alpha$ for α

$$m \cdot f_{mi} = -(1-p_i)\ln(1-p_i)\ln[-\ln(1-p_i)] = -\alpha \cdot f_{\alpha i} \tag{54}$$

Further

$$\alpha\beta \cdot f_{\beta i} = (1-p_i)\ln(1-p_i) \tag{55}$$

and

$$-\alpha\beta \cdot f_{\mu i} = (1-p_i)[-\ln(1-p_i)]^{1-\alpha} \tag{56}$$

which for the special case $\alpha = 1$ takes the simple form

$$-\beta \cdot f_{\mu i} = 1 - p_i \tag{57}$$

It may be noted that the quantities $\alpha \cdot f_{\alpha i}$, $m \cdot f_{mi}$, and $\alpha\beta \cdot f_{\beta i}$ are independent of all parameters and uniquely determined by $p_i = i/(N+1)$, while $\alpha\beta \cdot f_{\mu i}$ is a function of p_i and α.

Values of $\alpha \cdot f_{\alpha i}$ and $\alpha\beta \cdot f_{\beta i}$ are listed in Tables 1 and 2 for $p_i = 0.01(0.01)0.99$ and those of $\alpha \cdot f_{\mu i}$ in Table 3 for $\alpha = 0.1(0.1)1.0$ and $p_i = 0.01(0.01)0.99$.

For small and moderate sample sizes the sums AA, AB, etc. are easily computed. As $N \longrightarrow \infty$ the sums tend to the following values, where A, B, C, corresponds to α, β, μ:

$$\left.\begin{array}{lll}
(N+1)\alpha^2 \cdot AA & \longrightarrow 1.823680 & \\
(N+1)\alpha^2\beta^2 \cdot BB & \longrightarrow 1.0 & \\
(N+1)\alpha^2\beta \cdot AB & \longrightarrow 0.422784 & \\
(N+1)\alpha^2\beta^2 \cdot CC & \longrightarrow (1-\alpha)^2(1-2\alpha)!/(1-2\alpha) & \alpha < 0.5 \\
(N+1)\beta^2 \cdot CC & \longrightarrow \ln N & \alpha = 0.5 \\
(N+1)\beta^2 \cdot CC & \longrightarrow N^{2\alpha-1}/(2\alpha-1) & \alpha > 0.5 \\
(N+1)\alpha^2 \cdot AC & \longrightarrow (1-\alpha)![\ (1-\alpha)-\alpha/(1-\alpha)] & \alpha < 1.0 \\
(N+1)\alpha^2\beta^2 \cdot BC & \longrightarrow (1-\alpha)! &
\end{array}\right\} \tag{58}$$

7.2 One unknown parameter

The procedures will be illustrated by two examples, the first one assuming that β = unknown, α, μ known, $N = 9$.

Starting from the values of $\alpha\beta . f_{\beta i}$ in Table 2, we have

i	1 $\alpha\beta . f_{\beta i}$	2 $Op(f_{\beta i})$	3 $Op(f_{\beta i})/f_{\beta i}$	4 w_i
1	-.094824	-.011133	.117407	.8220
2	-.178515	-.012534	.070213	.4916
3	-.249672	-.014334	.057411	.4020
4	-.306495	-.016744	.054631	.3825
5	-.346574	-.020137	.058103	.4068
6	-.366516	-.025266	.068936	.4827
7	-.361192	-.033980	.094077	.6587
8	-.321888	-.052325	.162557	1.1382
9	-.230259	-.138630	.602061	4.2155
	-.325083	-.325083	1.285396	9.0000

The values $Op(f_{\beta i})$ are controlled by the relation $\Sigma Op(f_{\beta i}) = f_{\beta i} + f_{\beta N}$. Col.4 presents the weights normalized to $\Sigma w_i = N$. From Col.1 and 2 we have

$$\alpha^2\beta^2 . BB = \Sigma f_{\beta i} . Op(f_{\beta i}) = 0.089280$$

Hence by (28) and (37) the most efficient weighting yields

$$Var(\hat{\beta}) = 0.10182 \quad \alpha^2\beta^2$$

which is less than the lower bound of $Var(\hat{\beta}) = \alpha^2\beta^2/N = 0.11111\alpha^2\beta^2$. This is due to the fact that the scatter in the coefficients $f_{\beta i}$ has been neglected. The difference tends to zero as $N \to \infty$.

The second example is: μ = known, $\alpha = \beta = 1$, N arbitrary.

From $-f_{\mu i} = 1 - p_i$ it follows that all $Op(f_{\mu i}) = 0$ and consequently $w_i = 0$ for $i > 1$. Hence, only the smallest observation should be used for the estimation of μ. It is known that, if doing so, an efficiency of 100% is attained for any sample size N.

7.3 Two or more unknown parameters

The procedures will be illustrated by assuming
α, β unknown, μ known, $N = 9$

From the sums: $\alpha^2.AA = 0.12312$; $\alpha^2\beta^2.BB = 0.08928$; $\alpha^2\beta.AB = 0.02068$

the values of λ_α and λ_β are computed by use of (37) and then the weights $w_{\alpha i}$ and $w_{\beta i}$ which minimize $Var(\hat{\alpha})$ and $Var(\hat{\beta})$, respectively. These values are

$$Var(\hat{\alpha}) = 0.07748\,\alpha^2 \quad \text{and} \quad Var(\hat{\beta}) = 0.10598\,\alpha^2\beta^2$$

The corresponding lower bounds are

$$Var(\hat{\alpha}) = 0.06755\,\alpha^2 \quad \text{and} \quad Var(\hat{\beta}) = 0.123318$$

the latter actually being larger than that of the estimate, for reasons indicated above. Computed values are listed below.

i	$w_{\alpha i}$	$w_{\beta i}$	w_i
1	2.1642	2.4363	2.3002
2	7219	7988	7604
3	4937	5179	5058
4	4334	4233	4283
5	4677	4005	4341
6	8882	4299	6591
7	2167	5342	3754
8	6558	8357	7458
9	2.9583	2.6235	2.7909

where $w_i = (w_{\alpha i} + w_{\beta i})/2$.

If the weights $w_{\alpha i}$ are used for estimating β, we have

$$Var(\hat{\beta}) = 0.10680\,\alpha^2\beta^2$$

This small reduction in efficiency, being 0.8% only, motivates the use of a common set w_i for estimating both α and β.

Table 1. The coefficients $\alpha.f_{\alpha i}$

p_i	$\alpha.f_{\alpha i}$	p_i	$\alpha.f_{\alpha i}$	p_i	$\alpha.f_{\alpha i}$
0.01	0.045771	0.34	0.240847	0.67	-0.037740
0.02	0.077253	0.35	0.235810	0.68	-0.047594
0.03	0.103154	0.36	0.230439	0.69	-0.057370
0.04	0.125348	0.37	0.224749	0.70	-0.067047
0.05	0.144734	0.38	0.218751	0.71	-0.076605
0.06	0.161846	0.39	0.212458	0.72	-0.086024
0.07	0.177041	0.40	0.205881	0.73	-0.095280
0.08	0.190575	0.41	0.199033	0.74	-0.104348
0.09	0.202641	0.42	0.191925	0.75	-0.113203
0.10	0.213390	0.43	0.184568	0.76	-0.121815
0.11	0.222943	0.44	0.176974	0.77	-0.130154
0.12	0.231402	0.45	0.169152	0.78	-0.138186
0.13	0.238851	0.46	0.161115	0.79	-0.145876
0.14	0.245361	0.47	0.152873	0.80	-0.153181
0.15	0.250997	0.48	0.144436	0.81	-0.160060
0.16	0.255812	0.49	0.135816	0.82	-0.166461
0.17	0.259855	0.50	0.127024	0.83	-0.172330
0.18	0.263169	0.51	0.118069	0.84	-0.177607
0.19	0.265793	0.52	0.108964	0.85	-0.182219
0.20	0.267762	0.53	0.099719	0.86	-0.186089
0.21	0.269108	0.54	0.090344	0.87	-0.189123
0.22	0.269860	0.55	0.080853	0.88	-0.191216
0.23	0.270046	0.56	0.071255	0.89	-0.192239
0.24	0.269691	0.57	0.061563	0.90	-0.192043
0.25	0.268817	0.58	0.051788	0.91	-0.190444
0.26	0.267447	0.59	0.041944	0.92	-0.187213
0.27	0.265601	0.60	0.032042	0.93	-0.182062
0.28	0.263299	0.61	0.022095	0.94	-0.174611
0.29	0.260558	0.62	0.012116	0.95	-0.164344
0.30	0.257395	0.63	0.002121	0.96	-0.150519
0.31	0.253827	0.64	-0.007876	0.97	-0.131984
0.32	0.249870	0.65	-0.017865	0.98	-0.106725
0.33	0.245539	0.66	-0.027824	0.99	-0.070330

Table 2. The coefficients $\alpha\beta.f_{\beta i}$

p_i	$\alpha\beta.f_{\beta i}$	p_i	$\alpha\beta.f_{\beta i}$	p_i	$\alpha\beta.f_{\beta i}$
0.01	-0.009950	0.34	-0.274240	0.67	-0.365859
0.02	-0.019799	0.35	-0.280009	0.68	-0.364619
0.03	-0.029545	0.36	-0.285624	0.69	-0.363067
0.04	-0.039189	0.37	-0.291082	0.70	-0.361192
0.05	-0.048729	0.38	-0.296382	0.71	-0.358984
0.06	-0.058163	0.39	-0.301521	0.72	-0.356430
0.07	-0.067491	0.40	-0.306495	0.73	-0.353520
0.08	-0.076711	0.41	-0.311303	0.74	-0.350239
0.09	-0.085823	0.42	-0.315942	0.75	-0.346574
0.10	-0.094824	0.43	-0.320408	0.76	-0.342508
0.11	-0.103715	0.44	-0.324698	0.77	-0.338026
0.12	-0.112493	0.45	-0.328810	0.78	-0.333108
0.13	-0.121158	0.46	-0.332740	0.79	-0.327736
0.14	-0.129708	0.47	-0.336485	0.80	-0.321888
0.15	-0.138141	0.48	-0.340042	0.81	-0.315539
0.16	-0.146457	0.49	-0.343406	0.82	-0.308664
0.17	-0.154654	0.50	-0.346574	0.83	-0.301233
0.18	-0.162730	0.51	-0.349541	0.84	-0.293213
0.19	-0.170684	0.52	-0.352305	0.85	-0.284568
0.20	-0.178515	0.53	-0.354861	0.86	-0.275256
0.21	-0.186221	0.54	-0.357203	0.87	-0.265229
0.22	-0.193800	0.55	-0.359328	0.88	-0.254432
0.23	-0.201251	0.56	-0.361231	0.89	-0.242800
0.24	-0.208572	0.57	-0.362907	0.90	-0.230259
0.25	-0.215762	0.58	-0.364350	0.91	-0.216715
0.26	-0.222818	0.59	-0.365555	0.92	-0.202059
0.27	-0.229739	0.60	-0.366516	0.93	-0.186148
0.28	-0.236523	0.61	-0.367227	0.94	-0.168805
0.29	-0.243168	0.62	-0.367682	0.95	-0.149787
0.30	-0.249672	0.63	-0.367873	0.96	-0.128755
0.31	-0.256034	0.64	-0.367794	0.97	-0.105197
0.32	-0.262250	0.65	-0.367438	0.98	-0.078241
0.33	-0.268320	0.66	-0.366795	0.99	-0.046052

Table 3. The coefficients $\alpha\beta.f_{\mu i}$; $\alpha = 0.1$

p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$
0.01	-0.015762	0.34	-0.299414	0.67	-0.362104
0.02	-0.029248	0.35	-0.304611	0.68	-0.359890
0.03	-0.041891	0.36	-0.309623	0.69	-0.357375
0.04	-0.053961	0.37	-0.314448	0.70	-0.354549
0.05	-0.065581	0.38	-0.319085	0.71	-0.351404
0.06	-0.076824	0.39	-0.323533	0.72	-0.347931
0.07	-0.087734	0.40	-0.327791	0.73	-0.344119
0.08	-0.098345	0.41	-0.331857	0.74	-0.339958
0.09	-0.108679	0.42	-0.335729	0.75	-0.335436
0.10	-0.118755	0.43	-0.339407	0.76	-0.330541
0.11	-0.128587	0.44	-0.342887	0.77	-0.325258
0.12	-0.138186	0.45	-0.346168	0.78	-0.319572
0.13	-0.147560	0.46	-0.349248	0.79	-0.313468
0.14	-0.156718	0.47	-0.352125	0.80	-0.306928
0.15	-0.165666	0.48	-0.354797	0.81	-0.299932
0.16	-0.174408	0.49	-0.357259	0.82	-0.292459
0.17	-0.182950	0.50	-0.359512	0.83	-0.284483
0.18	-0.191294	0.51	-0.361550	0.84	-0.275980
0.19	-0.199445	0.52	-0.363372	0.85	-0.266917
0.20	-0.207403	0.53	-0.364974	0.86	-0.257262
0.21	-0.215173	0.54	-0.366353	0.87	-0.246975
0.22	-0.222755	0.55	-0.367505	0.88	-0.236011
0.23	-0.230151	0.56	-0.368428	0.89	-0.224318
0.24	-0.237362	0.57	-0.369116	0.90	-0.211833
0.25	-0.244390	0.58	-0.369566	0.91	-0.198484
0.26	-0.251234	0.59	-0.369774	0.92	-0.184178
0.27	-0.257895	0.60	-0.369734	0.93	-0.168804
0.28	-0.264374	0.61	-0.369443	0.94	-0.152217
0.29	-0.270671	0.62	-0.368896	0.95	-0.134222
0.30	-0.276786	0.63	-0.368085	0.96	-0.114550
0.31	-0.282717	0.64	-0.367007	0.97	-0.092793
0.32	-0.288467	0.65	-0.365656	0.98	-0.068264
0.33	-0.294032	0.66	-0.364023	0.99	-0.039530

Table 3. The coefficients $\alpha\beta.f_{\mu i}$; $\alpha = 0.2$

p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$
0.01	-0.024968	0.34	-0.326899	0.67	-0.358388
0.02	-0.043207	0.35	-0.331375	0.68	-0.355223
0.03	-0.059395	0.36	-0.335638	0.69	-0.351772
0.04	-0.074300	0.37	-0.339688	0.70	-0.348028
0.05	-0.088262	0.38	-0.343526	0.71	-0.343985
0.06	-0.101471	0.39	-0.347152	0.72	-0.339634
0.07	-0.114049	0.40	-0.350566	0.73	-0.334969
0.08	-0.126079	0.41	-0.353767	0.74	-0.329979
0.09	-0.137623	0.42	-0.356756	0.75	-0.324657
0.10	-0.148725	0.43	-0.359532	0.76	-0.318991
0.11	-0.159423	0.44	-0.362094	0.77	-0.312972
0.12	-0.169745	0.45	-0.364442	0.78	-0.306586
0.13	-0.179715	0.46	-0.366575	0.79	-0.299822
0.14	-0.189353	0.47	-0.368492	0.80	-0.292664
0.15	-0.198674	0.48	-0.370192	0.81	-0.285097
0.16	-0.207694	0.49	-0.371672	0.82	-0.277104
0.17	-0.216423	0.50	-0.372933	0.83	-0.268665
0.18	-0.224873	0.51	-0.373971	0.84	-0.259759
0.19	-0.233051	0.52	-0.374786	0.85	-0.250361
0.20	-0.240967	0.53	-0.375375	0.86	-0.240445
0.21	-0.248627	0.54	-0.375737	0.87	-0.229978
0.22	-0.256036	0.55	-0.375868	0.88	-0.218924
0.23	-0.263201	0.56	-0.375767	0.89	-0.207242
0.24	-0.270127	0.57	-0.375431	0.90	-0.194883
0.25	-0.276816	0.58	-0.374857	0.91	-0.181786
0.26	-0.283274	0.59	-0.374041	0.92	-0.167880
0.27	-0.289502	0.60	-0.372981	0.93	-0.153076
0.28	-0.295505	0.61	-0.371673	0.94	-0.137258
0.29	-0.301285	0.62	-0.370113	0.95	-0.120274
0.30	-0.306843	0.63	-0.368298	0.96	-0.101912
0.31	-0.312182	0.64	-0.366222	0.97	-0.081852
0.32	-0.317303	0.65	-0.363882	0.98	-0.059560
0.33	-0.322209	0.66	-0.361272	0.99	-0.033931

Table 3. The coefficients $\alpha\beta.f_{\mu i}$; $\alpha = 0.3$

p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$
0.01	-0.039551	0.34	-0.356907	0.67	-0.354710
0.02	-0.063828	0.35	-0.360491	0.68	-0.350617
0.03	-0.084212	0.36	-0.363840	0.69	-0.346257
0.04	-0.102305	0.37	-0.366955	0.70	-0.341628
0.05	-0.118786	0.38	-0.369840	0.71	-0.336722
0.06	-0.134026	0.39	-0.372495	0.72	-0.331535
0.07	-0.148257	0.40	-0.374923	0.73	-0.326061
0.08	-0.161636	0.41	-0.377124	0.74	-0.320293
0.09	-0.174275	0.42	-0.379100	0.75	-0.314224
0.10	-0.186259	0.43	-0.380851	0.76	-0.307846
0.11	-0.197654	0.44	-0.382378	0.77	-0.301150
0.12	-0.208513	0.45	-0.383681	0.78	-0.294128
0.13	-0.218878	0.46	-0.384762	0.79	-0.286769
0.14	-0.228784	0.47	-0.385620	0.80	-0.279063
0.15	-0.238260	0.48	-0.386255	0.81	-0.270996
0.16	-0.247332	0.49	-0.386666	0.82	-0.262556
0.17	-0.256021	0.50	-0.386855	0.83	-0.253727
0.18	-0.264345	0.51	-0.386819	0.84	-0.244492
0.19	-0.272321	0.52	-0.386559	0.85	-0.234832
0.20	-0.279962	0.53	-0.386073	0.86	-0.224726
0.21	-0.287281	0.54	-0.385361	0.87	-0.214150
0.22	-0.294290	0.55	-0.384422	0.88	-0.203074
0.23	-0.300998	0.56	-0.383253	0.89	-0.191466
0.24	-0.307414	0.57	-0.381854	0.90	-0.179288
0.25	-0.313545	0.58	-0.380223	0.91	-0.166493
0.26	-0.319400	0.59	-0.378357	0.92	-0.153025
0.27	-0.324983	0.60	-0.376256	0.93	-0.138813
0.28	-0.330302	0.61	-0.373916	0.94	-0.123770
0.29	-0.335361	0.62	-0.371335	0.95	-0.107776
0.30	-0.340164	0.63	-0.368510	0.96	-0.090668
0.31	-0.344717	0.64	-0.365439	0.97	-0.072200
0.32	-0.349023	0.65	-0.362117	0.98	-0.051965
0.33	-0.353085	0.66	-0.358542	0.99	-0.029126

Table 3. The coefficients $\alpha\beta.f_{\mu i}$; $\alpha = 0.4$

p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$
0.01	-0.062653	0.34	-0.389669	0.67	-0.351070
0.02	-0.094291	0.35	-0.392165	0.68	-0.346070
0.03	-0.119400	0.36	-0.394411	0.69	-0.340829
0.04	-0.140867	0.37	-0.396411	0.70	-0.335345
0.05	-0.159867	0.38	-0.398169	0.71	-0.329613
0.06	-0.177027	0.39	-0.399689	0.72	-0.323630
0.07	-0.192726	0.40	-0.400973	0.73	-0.317391
0.08	-0.207219	0.41	-0.402023	0.74	-0.310891
0.09	-0.220688	0.42	-0.402843	0.75	-0.304126
0.10	-0.233265	0.43	-0.403433	0.76	-0.297089
0.11	-0.245054	0.44	-0.403797	0.77	-0.289775
0.12	-0.256135	0.45	-0.403936	0.78	-0.282176
0.13	-0.266575	0.46	-0.403851	0.79	-0.274285
0.14	-0.276426	0.47	-0.403543	0.80	-0.266094
0.15	-0.285734	0.48	-0.403015	0.81	-0.257593
0.16	-0.294535	0.49	-0.402265	0.82	-0.248771
0.17	-0.302864	0.50	-0.401296	0.83	-0.239619
0.18	-0.310746	0.51	-0.400108	0.84	-0.230122
0.19	-0.318207	0.52	-0.398702	0.85	-0.220266
0.20	-0.325267	0.53	-0.397076	0.86	-0.210036
0.21	-0.331946	0.54	-0.395232	0.87	-0.199411
0.22	-0.338259	0.55	-0.393170	0.88	-0.188372
0.23	-0.344222	0.56	-0.390888	0.89	-0.176891
0.24	-0.349848	0.57	-0.388387	0.90	-0.164941
0.25	-0.355148	0.58	-0.385666	0.91	-0.152486
0.26	-0.360133	0.59	-0.382724	0.92	-0.139483
0.27	-0.364813	0.60	-0.379560	0.93	-0.125880
0.28	-0.369196	0.61	-0.376172	0.94	-0.111607
0.29	-0.373291	0.62	-0.372561	0.95	-0.096577
0.30	-0.377104	0.63	-0.368723	0.96	-0.080665
0.31	-0.380643	0.64	-0.364657	0.97	-0.063687
0.32	-0.383913	0.65	-0.360361	0.98	-0.045339
0.33	-0.386920	0.66	-0.355833	0.99	-0.025001

Table 3. The coefficients $\alpha\beta.f_{\mu i}$; $\alpha = 0.5$

p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$
0.01	-0.099249	0.34	-0.425439	0.67	-0.347467
0.02	-0.139294	0.35	-0.426621	0.68	-0.341582
0.03	-0.169290	0.36	-0.427550	0.69	-0.335486
0.04	-0.193963	0.37	-0.428231	0.70	-0.329177
0.05	-0.215156	0.38	-0.428669	0.71	-0.322653
0.06	-0.233823	0.39	-0.428868	0.72	-0.315912
0.07	-0.250532	0.40	-0.428832	0.73	-0.308951
0.08	-0.265658	0.41	-0.428566	0.74	-0.301765
0.09	-0.279461	0.42	-0.428073	0.75	-0.294353
0.10	-0.292134	0.43	-0.427355	0.76	-0.286709
0.11	-0.303820	0.44	-0.426417	0.77	-0.278830
0.12	-0.314633	0.45	-0.425260	0.78	-0.270710
0.13	-0.324665	0.46	-0.423887	0.79	-0.262344
0.14	-0.333989	0.47	-0.422300	0.80	-0.253727
0.15	-0.342666	0.48	-0.420502	0.81	-0.244852
0.16	-0.350747	0.49	-0.418494	0.82	-0.235711
0.17	-0.358277	0.50	-0.416277	0.83	-0.226295
0.18	-0.365292	0.51	-0.413854	0.84	-0.216597
0.19	-0.371825	0.52	-0.411226	0.85	-0.206604
0.20	-0.377905	0.53	-0.408393	0.86	-0.196306
0.21	-0.383555	0.54	-0.405356	0.87	-0.185687
0.22	-0.388798	0.55	-0.402117	0.88	-0.174734
0.23	-0.393654	0.56	-0.398675	0.89	-0.163426
0.24	-0.398139	0.57	-0.395032	0.90	-0.151743
0.25	-0.402270	0.58	-0.391187	0.91	-0.139658
0.26	-0.406061	0.59	-0.387140	0.92	-0.127141
0.27	-0.409523	0.60	-0.382892	0.93	-0.114151
0.28	-0.412670	0.61	-0.378442	0.94	-0.100640
0.29	-0.415511	0.62	-0.373790	0.95	-0.086541
0.30	-0.418056	0.63	-0.368935	0.96	-0.071765
0.31	-0.420313	0.64	-0.363876	0.97	-0.056178
0.32	-0.422292	0.65	-0.358613	0.98	-0.039558
0.33	-0.423998	0.66	-0.353144	0.99	-0.021460

Table 3. The coefficients $\alpha\beta.f_{\mu i}$; $\alpha = 0.6$

p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$
0.01	-0.157220	0.34	-0.464493	0.67	-0.343901
0.02	-0.205774	0.35	-0.464106	0.68	-0.337152
0.03	-0.240026	0.36	-0.463474	0.69	-0.330226
0.04	-0.267072	0.37	-0.462605	0.70	-0.323123
0.05	-0.289566	0.38	-0.461504	0.71	-0.315841
0.06	-0.308841	0.39	-0.460177	0.72	-0.308379
0.07	-0.325677	0.40	-0.458628	0.73	-0.300735
0.08	-0.340577	0.41	-0.456862	0.74	-0.292907
0.09	-0.353888	0.42	-0.454883	0.75	-0.284893
0.10	-0.365859	0.43	-0.452695	0.76	-0.276691
0.11	-0.376678	0.44	-0.450303	0.77	-0.268298
0.12	-0.386492	0.45	-0.447709	0.78	-0.259710
0.13	-0.395414	0.46	-0.444917	0.79	-0.250924
0.14	-0.403539	0.47	-0.441929	0.80	-0.241936
0.15	-0.410942	0.48	-0.438748	0.81	-0.232741
0.16	-0.417687	0.49	-0.435377	0.82	-0.223336
0.17	-0.423829	0.50	-0.431817	0.83	-0.213713
0.18	-0.429413	0.51	-0.428072	0.84	-0.203866
0.19	-0.434478	0.52	-0.424143	0.85	-0.193789
0.20	-0.439060	0.53	-0.420032	0.86	-0.183473
0.21	-0.443187	0.54	-0.415739	0.87	-0.172908
0.22	-0.446887	0.55	-0.411267	0.88	-0.162083
0.23	-0.450183	0.56	-0.406617	0.89	-0.150986
0.24	-0.453096	0.57	-0.401790	0.90	-0.139600
0.25	-0.455645	0.58	-0.396787	0.91	-0.127909
0.26	-0.457846	0.59	-0.391608	0.92	-0.115890
0.27	-0.459714	0.60	-0.386254	0.93	-0.103515
0.28	-0.461263	0.61	-0.380726	0.94	-0.090750
0.29	-0.462506	0.62	-0.375024	0.95	-0.077548
0.30	-0.463454	0.63	-0.369148	0.96	-0.063847
0.31	-0.464118	0.64	-0.363098	0.97	-0.049554
0.32	-0.464507	0.65	-0.356874	0.98	-0.034514
0.33	-0.464629	0.66	-0.350475	0.99	-0.018421

Table 3. The coefficients $\alpha\beta.f_{\mu i}$; $\alpha = 0.7$

p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$
0.01	-0.249052	0.34	-0.507131	0.67	-0.340372
0.02	-0.303983	0.35	-0.504883	0.68	-0.332780
0.03	-0.340320	0.36	-0.502417	0.69	-0.325049
0.04	-0.367739	0.37	-0.499739	0.70	-0.317180
0.05	-0.389710	0.38	-0.496855	0.71	-0.309173
0.06	-0.407928	0.39	-0.493772	0.72	-0.301025
0.07	-0.423361	0.40	-0.490493	0.73	-0.292738
0.08	-0.436625	0.41	-0.487025	0.74	-0.284309
0.09	-0.448136	0.42	-0.483372	0.75	-0.275738
0.10	-0.458190	0.43	-0.479538	0.76	-0.267023
0.11	-0.467009	0.44	-0.475528	0.77	-0.258163
0.12	-0.474762	0.45	-0.471344	0.78	-0.249156
0.13	-0.481581	0.46	-0.466990	0.79	-0.240000
0.14	-0.487572	0.47	-0.462469	0.80	-0.230692
0.15	-0.492822	0.48	-0.457786	0.81	-0.221230
0.16	-0.497403	0.49	-0.452941	0.82	-0.211610
0.17	-0.501375	0.50	-0.447938	0.83	-0.201830
0.18	-0.504789	0.51	-0.442779	0.84	-0.191884
0.19	-0.507689	0.52	-0.437466	0.85	-0.181769
0.20	-0.510112	0.53	-0.432002	0.86	-0.171479
0.21	-0.512091	0.54	-0.426388	0.87	-0.161008
0.22	-0.513656	0.55	-0.420626	0.88	-0.150348
0.23	-0.514831	0.56	-0.414718	0.89	-0.139492
0.24	-0.515640	0.57	-0.408664	0.90	-0.128430
0.25	-0.516101	0.58	-0.402467	0.91	-0.117149
0.26	-0.516235	0.59	-0.396127	0.92	-0.105635
0.27	-0.516055	0.60	-0.389646	0.93	-0.093870
0.28	-0.515578	0.61	-0.383024	0.94	-0.081832
0.29	-0.514817	0.62	-0.376262	0.95	-0.069490
0.30	-0.513783	0.63	-0.369361	0.96	-0.056803
0.31	-0.512488	0.64	-0.362321	0.97	-0.043711
0.32	-0.510941	0.65	-0.355143	0.98	-0.030113
0.33	-0.509153	0.66	-0.347826	0.99	-0.015812

Table 3. The coefficients $\alpha\beta.f_{\mu i}$; $\alpha = 0.8$

p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$
0.01	-0.394522	0.34	-0.553683	0.67	-0.336879
0.02	-0.449064	0.35	-0.549244	0.68	-0.328464
0.03	-0.482520	0.36	-0.544632	0.69	-0.319953
0.04	-0.506349	0.37	-0.539853	0.70	-0.311347
0.05	-0.524488	0.38	-0.534914	0.71	-0.302645
0.06	-0.538805	0.39	-0.529819	0.72	-0.293847
0.07	-0.550345	0.40	-0.524573	0.73	-0.284953
0.08	-0.559760	0.41	-0.519180	0.74	-0.275964
0.09	-0.567484	0.42	-0.513646	0.75	-0.266877
0.10	-0.573823	0.43	-0.507973	0.76	-0.257693
0.11	-0.579002	0.44	-0.502165	0.77	-0.248412
0.12	-0.583192	0.45	-0.496226	0.78	-0.239032
0.13	-0.586524	0.46	-0.490158	0.79	-0.229552
0.14	-0.589104	0.47	-0.483965	0.80	-0.219971
0.15	-0.591017	0.48	-0.477649	0.81	-0.210288
0.16	-0.592332	0.49	-0.471213	0.82	-0.200500
0.17	-0.593109	0.50	-0.464660	0.83	-0.190607
0.18	-0.593396	0.51	-0.457991	0.84	-0.180606
0.19	-0.593235	0.52	-0.451208	0.85	-0.170495
0.20	-0.592662	0.53	-0.444314	0.86	-0.160269
0.21	-0.591708	0.54	-0.437310	0.87	-0.149927
0.22	-0.590400	0.55	-0.430198	0.88	-0.139463
0.23	-0.588762	0.56	-0.422979	0.89	-0.128874
0.24	-0.586816	0.57	-0.415656	0.90	-0.118153
0.25	-0.584580	0.58	-0.408228	0.91	-0.107293
0.26	-0.582070	0.59	-0.400699	0.92	-0.096287
0.27	-0.579302	0.60	-0.393067	0.93	-0.085124
0.28	-0.576289	0.61	-0.385335	0.94	-0.073790
0.29	-0.573044	0.62	-0.377504	0.95	-0.062269
0.30	-0.569577	0.63	-0.369574	0.96	-0.050536
0.31	-0.565899	0.64	-0.361546	0.97	-0.038557
0.32	-0.562018	0.65	-0.353420	0.98	-0.026273
0.33	-0.557944	0.66	-0.345198	0.99	-0.013572

Table 3. The coefficients $\alpha\beta.f_{\mu i}$; $\alpha = 0.9$

p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$
0.01	-0.624961	0.34	-0.604509	0.67	-0.333422
0.02	-0.663387	0.35	-0.597502	0.68	-0.324204
0.03	-0.684138	0.36	-0.590393	0.69	-0.314937
0.04	-0.697205	0.37	-0.583187	0.70	-0.305621
0.05	-0.705878	0.38	-0.575888	0.71	-0.296255
0.06	-0.711672	0.39	-0.568498	0.72	-0.286840
0.07	-0.715417	0.40	-0.561020	0.73	-0.277376
0.08	-0.717620	0.41	-0.553459	0.74	-0.267863
0.09	-0.718617	0.42	-0.545816	0.75	-0.258301
0.10	-0.718638	0.43	-0.538093	0.76	-0.248689
0.11	-0.717852	0.44	-0.530295	0.77	-0.239029
0.12	-0.716386	0.45	-0.522421	0.78	-0.229319
0.13	-0.714336	0.46	-0.514476	0.79	-0.219558
0.14	-0.711779	0.47	-0.506460	0.80	-0.209748
0.15	-0.708776	0.48	-0.498375	0.81	-0.199887
0.16	-0.705379	0.49	-0.490223	0.82	-0.189974
0.17	-0.701627	0.50	-0.482006	0.83	-0.180009
0.18	-0.697556	0.51	-0.473725	0.84	-0.169991
0.19	-0.693196	0.52	-0.465381	0.85	-0.159919
0.20	-0.688570	0.53	-0.456977	0.86	-0.149792
0.21	-0.683702	0.54	-0.448512	0.87	-0.139608
0.22	-0.678610	0.55	-0.439988	0.88	-0.129366
0.23	-0.673311	0.56	-0.431406	0.89	-0.119064
0.24	-0.667818	0.57	-0.422767	0.90	-0.108698
0.25	-0.662144	0.58	-0.414072	0.91	-0.098267
0.26	-0.656302	0.59	-0.405323	0.92	-0.087767
0.27	-0.650300	0.60	-0.396518	0.93	-0.077192
0.28	-0.644149	0.61	-0.387661	0.94	-0.066539
0.29	-0.637857	0.62	-0.378750	0.95	-0.055798
0.30	-0.631430	0.63	-0.369787	0.96	-0.044961
0.31	-0.624876	0.64	-0.360772	0.97	-0.034010
0.32	-0.618201	0.65	-0.351706	0.98	-0.022923
0.33	-0.611410	0.66	-0.342589	0.99	-0.011650

Table 3. The coefficients $\alpha\beta.f_{\mu i}$; $\alpha = 1.0$

p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$	p_i	$\alpha\beta.f_{\mu i}$
0.01	-0.990000	0.34	-0.660000	0.67	-0.330000
0.02	-0.980000	0.35	-0.650000	0.68	-0.320000
0.03	-0.970000	0.36	-0.640000	0.69	-0.310000
0.04	-0.960000	0.37	-0.630000	0.70	-0.300000
0.05	-0.950000	0.38	-0.620000	0.71	-0.290000
0.06	-0.940000	0.39	-0.610000	0.72	-0.280000
0.07	-0.930000	0.40	-0.600000	0.73	-0.270000
0.08	-0.920000	0.41	-0.590000	0.74	-0.260000
0.09	-0.910000	0.42	-0.580000	0.75	-0.250000
0.10	-0.900000	0.43	-0.570000	0.76	-0.240000
0.11	-0.890000	0.44	-0.560000	0.77	-0.230000
0.12	-0.880000	0.45	-0.550000	0.78	-0.220000
0.13	-0.870000	0.46	-0.540000	0.79	-0.210000
0.14	-0.860000	0.47	-0.530000	0.80	-0.200000
0.15	-0.850000	0.48	-0.520000	0.81	-0.190000
0.16	-0.840000	0.49	-0.510000	0.82	-0.180000
0.17	-0.830000	0.50	-0.500000	0.83	-0.170000
0.18	-0.820000	0.51	-0.490000	0.84	-0.160000
0.19	-0.810000	0.52	-0.480000	0.85	-0.150000
0.20	-0.800000	0.53	-0.470000	0.86	-0.140000
0.21	-0.790000	0.54	-0.460000	0.87	-0.130000
0.22	-0.780000	0.55	-0.450000	0.88	-0.120000
0.23	-0.770000	0.56	-0.440000	0.89	-0.110000
0.24	-0.760000	0.57	-0.430000	0.90	-0.100000
0.25	-0.750000	0.58	-0.420000	0.91	-0.090000
0.26	-0.740000	0.59	-0.410000	0.92	-0.080000
0.27	-0.730000	0.60	-0.400000	0.93	-0.070000
0.28	-0.720000	0.61	-0.390000	0.94	-0.060000
0.29	-0.710000	0.62	-0.380000	0.95	-0.050000
0.30	-0.700000	0.63	-0.370000	0.96	-0.040000
0.31	-0.690000	0.64	-0.360000	0.97	-0.030000
0.32	-0.680000	0.65	-0.350000	0.98	-0.020000
0.33	-0.670000	0.66	-0.340000	0.99	-0.010000

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13. ABSTRACT

The method presented is applicable to complete, censored, or truncated samples and to grouped data drawn from any population having a continuous distribution function, simple or composed, involving an arbitrary number of unknown parameters. The estimates are consistent and asymptotically efficient (in some cases for any sample size) and easily determined by use of a versatile computer program. The efficiency can be stated for any individual case, even when only a part of the sample is used for the estimation. Two criteria of goodness-of-fit, which complete each other, makes it possible to decide whether the fit attained is acceptable or not.

Two of its many applications may be mentioned: the evaluation of data from bending and torsional tests on brittle materials, a problem up-to-now not quite satisfactorily solved due to the complicated distribution functions arising; and the analysis of bimodal fatigue-life distributions.

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