A CRITERION FOR THE ACCEPTABILITY OF ASSUMED DISTRIBUTIONS

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FOREWORD

This report was prepare by Prof. Dr. Waloddi Weibull, La Rosiaz, Lausanne, Switzerland under USAF Contract No. AF 61(052)-943. This contract was initiated under Project No. 7351, "Metallic Materials", Task No. 735106, "Behavior of Metals". The contract was administered by the European Office, Office of Aerospace Research. The work was monitored by the Metals and Ceramics Division, Air Force Materials Laboratory, Air Force Systems Command, under the direction of Mr. W. J. Trapp.

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This technical report has been reviewed and is approved.

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ABSTRACT

The usual way to solve the fundamental problem of deciding whether an assumed distribution function is acceptable or not, consists - if at all done - in estimating the parameters and checking the attained goodness of fit by some accepted criterion, in most cases the Chi-square test. In this way, the decision depends not only on the assumed function, so it may happen that an acceptable function may be rejected on the basis of results from poor estimating or fitting procedures.

The purpose of this research was to find a criterion which eliminates such fatalities and depends entirely on the assumed function alone. Such a criterion, based on the "number-of-runs", has been proposed. The properties of this statistic and its usefulness as a measure of departure from the true distribution have been demonstrated. The concept "maximum number of runs" (MAXNOR) of a given sample and methods for its ascertaining have been introduced. Its use as a criterion for deciding whether the assumed function is acceptable or not has been studied by applying it to data from tests on strength of brittle materials, fatigue life of aluminum alloys, etc.

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1. Introduction

Any statistical analysis of a set of observations is based on a - tacitly or expressedly - assumed distribution of the observations. Even a procedure as simple as taking the arithmetic mean of a number of observed values has no sense, if the underlying distribution is unknown. Consequently, the problem of deciding whether the assumed distribution function is acceptable or not is a fundamental problem.

The usual way of solving this problem consists in estimating from the sample, by some accepted method, the parameters of the assumed function and then testing the goodness-of-fit by use of some criterion, the mostly preferred one being the χ^2 test.

It is obvious that the decision will, in this way, depend not only on the assumed distribution but also on the estimation method and the criterion used, so it may easily happen that an acceptable function may be rejected as a result of a poor fitting procedure.

It has been the scope of this investigation to find a criterion for the acceptability of an assumed distribution of the population, from which the sample is drawn, which depends entirely on this distribution and is completely independent of any estimation procedure and the attained fit. Such a criterion, based on the statistic "number-of-runs", will now be demonstrated.

This criterion provides a tool not only for deciding whether an assumed distribution function is acceptable or not but also which of several competing functions is the better one, that is, has the least departure from the true but unknown distribution function.

2. Properties of the statistic number-of-runs

A run is, in this particular case, a sequence of data points lying on the same side of a curve. Let us now assume that this curve goes through the medians of the ordered observations. The sample of seven data points located as demonstrated in Fig.1 in relation to this curve thus yields four runs. It is evident that the probability of anyone of this points to fall above the curve, denoted by the sign +, is $p_i = 50\%$, and to fall below the curve, −, is $1 - p_i = 50\%$.

It should be noted that the number-of-runs (NOR), will be the same, even if other scales are used, for instance, those which present the median curve as a straight line, that is, using normal probability paper for normal distributions, etc. The median percentage points $\check{P}_i$ are independent of the actual distribution and are uniquely determined by the order number i and the sample size N. Tables of $\check{P}_i$ are given by Johnson [1] for $N = 1(1)20$ and by Weibull [2] for $N = 1(1)25$ where also corresponding percentiles $\check{y}_i = \log[-\ln(1-\check{P}_i)]$ and a very good approximation, introduced by Benard & Bos-Levenbach [3]

$$\check{P}_i = (i - 0.3)/(N + 0.4) \tag{1}$$

is examined.

Let us now further assume that the individual deviations from the curve are independent of each other. Then any possible arrangement of J signs, + or −, arbitrarily selected from the N available, will have the same probability $P = 1/2^J$, and the probability $f(u)$ that there will be u runs is proportional to the number of arrangements yielding this number of runs. Obviously, $f(u)$ is the discrete frequency (density) function of the statistic $U = NOR$.

In Table 1 all possible arrangements of J signs having $u = 1, 2, 3, \ldots$ runs are presented for $J = 1(1)5$. It is easily seen that the number of arrangements having u runs are proportional to the binomial coefficients C_{u-1}^{J-1} and that

$$f(u) = C_{u-1}^{J-1} / 2^{J-1} \quad (u = 1, 2, \ldots J) \tag{2}$$

The Cdf (cumulative distribution function) $F(u)$ of U is obtained by summing the values of $f(u)$. Hence

$$F(u) = \sum_{i=1}^{u} f(i) = \Sigma\, C_{i-1}^{J-1} / 2^{J-1} \tag{3}$$

The values of $F(u)$ have been computed for $J = 6(1)50(10)100$ and are presented in Table 2. For small values of J, equ.(3) has been used, starting from $u = 1$ and going upwards. For large values of J this procedure becomes tedious and it is more convenient to start from the middle of the sample, that is, from $u = J/2$ and going downwards. Values for $u > J/2$ may be computed by

$$F(u) = 100\% - F(J - u) \tag{4}$$

Since for even values of J

$$F(J/2) = 0.5 \tag{5}$$

it follows from (3) that

$$\left.\begin{aligned} F(J/2-1) &= 0.5 - C^{J-1}_{J/2-1}/2^{J-1} \\ F(J/2-2) &= 0.5 - (C^{J-1}_{J/2-1} + C^{J-1}_{J/2-2})/2^{J-1} \\ &\text{etc.} \end{aligned}\right\} \tag{6}$$

where by definition,

$$C^{J-1}_{u-1} = (J-1)!/(J/2-1)!(J/2)! = \Gamma(J)/\,\Gamma(J/2)\Gamma(J/2+1) \tag{7}$$

Knowing that the Gamma functions Γ become extremely large even for moderate arguments, its more convenient to use their logaritms. The values of $\log \Gamma(n)$ may be found in Table 6.4 of Handbook of Mathematical Functions, NBS Appl.Math. Series 55, 1964, for $n = 1(1)101$. For larger values of n the logarithms are equal to

$$\ln \Gamma(n) = (n - 0.5) \ln n - n + f_2(n) \tag{8}$$

where $f_2(n)$ is given in Table 6.5 (l.c.). For large arguments $f_2(n)$ does not vary much, for instance,

$$f_2(100) = 0.91977\,;\quad f_2(1000) = 0.91902\,;\quad f_2(\infty) = 0.91894$$

For moderate values of J the following rule of thumb may be used

$$F(0.5J) = 50\% \ ; \ F(0.4J) \ ; \ F(0.3J) = 1\%$$

It is an important fact that, for a given probability of F(u), the number of runs increases in proportion to the number of data points J, so it may be postulated that <u>for large samples, the true median curve will yield a large number of runs, which increases with the sample size.</u>

3. The number-of-runs as a measure of departure from the true median curve

Let us suppose that the true and the assumed median curves have the shape as indicated in Fig.2 with three intersection points. (If the assumed function involves c parameters, it is always possible to let the curves intersect in c points.) The departure of the assumed curve from the true one is completely defined by the differences

$$\Delta P_i = P_i - \breve{P}_i \tag{9}$$

The probability p_i' that the i:th data point falls above the assumed curve is uniquely determined by ΔP_i, i and N. As demonstrated in an AFML Report by Weibull [2], the percentage point P_i of the i:th order statistic has a Cdf which is

$$F(P_i) = i\, C_i^N \int_0^{P_i} P^{i-1}(1-P)^{N-i}\,dP \tag{10}$$

with the median defined by

$$i\, C_i^N \int_0^{\breve{P}_i} P^{i-1}(1-P)^{N-i}\,dP = 0.5 \tag{11}$$

This function $F(P_i)$ does not depend on the distribution of the observations and is uniquely determined by i and N. Five percentage points of the function $F(P_i)$ are listed in Table 1 (l.c.) for $N = 1(1)25$. The complete function is easily computed for $N = 1(1)50$ by use of tables prepared by Pearson [4], where the incomplete Beta-function is defined by

$$B_x(p,q) = \int_0^x x^{p-1}(1-x)^{q-1}\,dx \tag{12}$$

Dividing this function by the complete Beta-function $B(p,q) = B_1(p,q)$ the tabledfunction

$$I_x(p,q) = B_x(p,q)/B(p,q) \tag{13}$$

is obtained.

Comparing (10) and (12) it is easily found that

$$\left.\begin{aligned} F(P_i) &= I_x(p,q) \\ x = P_i \;;\; p = i \;&;\; q = N \end{aligned}\right\} \tag{14}$$

These curves for $i/N = 3/5$ and $i/N = 13/25$ are indicated in Fig.2. From this graph it may be seen that

$$p_i' = 1 - F(\breve{P}_i + \Delta P_i) \tag{15}$$

Since $F(P_i)$ and P_i are given by (10) and (11) for any (i,N), the probability p_i' can be computed for any given difference ΔP_i.

For example, from the graph in Fig.2 we have for $\Delta P_i = 5\%$, $i/N = 3/5$ and $13/25$ the probabilities

$p_i' = 57\%$ and 66%, respectively.

If, for any given sample J/N the J differences ΔP_i are known, then all the probabilities p_i' can be computed by use of (15).

Considering that the probability of the sequence ++−+ is equal to $p_1 \cdot p_2 \cdot (1-p_3) \cdot p_4$, etc., the Cdf $F(u)$ can be computed for any set of p_i'. This is an easy task for small values of u, for instance, for $u = 1$ we have

$$f(1) = F(1) = p_1' \cdot p_2' \dots p_J' + (1-p_1')(1-p_2') \dots (1-p_J')$$

but for large u it requires extensive, but principally easy computations.

In order to get an idea of the effect of the set p_i' on the number-of-runs, two numerical examples have been carried out. Suppose that we take out 5, alternatively 10, data points located between any of two consecutive intersection points, having all of them $p_i' > 0.5$, then, in order to simplify the calculations, the average of their individual p_i' has been attributed to each of them.

The results are presented in Table 3 for various p_i. It is readily seen that the probability of one single run increases from 6.25% (0.20%) to 100% as p_i increases from 0.5 to 1.0. At the same time the expected number-of-runs decreases from 3.0 (5.5) to 1.0.

Furthermore, since it follows from equ.(15) that for any given ΔP_i, differing from zero, $p_i' \rightarrow 1$ as $N \rightarrow \infty$, there will always exist a sample size N large enough to reduce the number-of-runs to one for each region between two intersection points, so it can be postulated that for large samples, the assumed median curve, if differing from the true one, will yield a small number-of-runs, which decreases with the sample size.

The fact that any departure from the true median distribution implies a reduction in number-of-runs, increasing with the magnitude of departure, motivates the use of the number-of-runs as a measure of this departure.

4. The maximum number of runs (MAXNOR) of a given sample

It is evident that, for any given sample J/N, there exists a maximum value of the number-of-runs, for any arbitrarily assumed median distribution function. The importance of this number, called the MAXNOR, is explained by the fact that, if this number has been determined and it corresponds to a probability, taken, say, from Table 2, which is smaller than an accepted level of significance, then it can be concluded that there does not exist any set of parameters of the assumed distribution, estimated by any method whatsoever, such that it will yield an acceptable number-of-runs. In other words, the true median curve cannot be closely enough reproduced by the assumed distribution function, which consequently has to be rejected.

The MAXNOR can be determined in the following way. Let the curve in Fig.3 correspond to a set of parameters of the assumed function, which yields the MAXNOR of the given sample of eight data points. If the data points are connected by a broken line, as indicated, then the number-of-runs is always one more than the number of crossings between this line and the assumed curve. In Fig. 3 there are 4 crossings and 5 runs.

If now the function involves, say, three parameters and these parameters are continuously varied, then there will be no change in the number-of-runs until the thus deformed curve passes through an "effective" point. The term "effective" implies that two curves, located at the opposite side of a point, will yield different number-of-runs. Otherwise the point is "neglectable". In Fig.3, point 5 is a neglectable and point 6 an effective one.

It will always be possible to vary the three parameters of the assumed function in such a way that the curve "touches" three points, not necessarily effective, without having passed over an effective point. (The term "touch" implies that the distance between the point and the curve is infinitesimal. The differences between the parameters of a curve going through three points and touching them will also be infinitesimal). It is obvious that also this touching curve yields the MAXNOR, so it may be concluded that, if the assumed distribution involves c parameters, there will always exist at least one set of c data points, such that a curve going through them will yield the MAXNOR. This conclusion indicates a method for its determination.

5. Methods for ascertaining the MAXNOR

From the preceding it follows that we have to count the number-of-runs for the curves going through all possible combinations of c data points and to identify the maximum number attained with the MAXNOR.

Observing that the number-of-combinations (NOC) for J points is

$$NOC = J(J-1)(\ldots(J-c)/1.2\ldots c$$

this number becomes very large even for moderate J.

For instance, the values of NOC are

J	c = 2	c = 3
10	45	120
25	300	2,300
50	1,225	19,600
100	4,950	161,700

These large numbers make the use of a computer indispensable, but even so, a reduction of the numbers is highly desirable. One such way is to suppress the neglectable data points. It should, however, be mentioned that, whether a point is neglectable or not, depends to some extent on the assumed distribution, so, in some cases, a preliminary, graphical analysis is recommended. For the two-parametric functions it may even substitute the computer program as will now be demonstrated.

5.1 Graphical methods

For two-parametric functions, scales can be used which will present the assumed Cdf as a straight line. The data points are obtained by plotting the observations x_i or their logarithms against appropriate plotting positions which are:

For the normal distribution:

$$\phi^{-1}(\breve{P}_i) \tag{16}$$

where ϕ is the normal Cdf. This is identical with plotting x_i against $\breve{P}_i$ on a normal probability paper.

For the Weibull distribution (α known)

$$\breve{z}_i = [-\ln(1-\breve{P}_i)]^{\alpha} \tag{17}$$

where $\alpha = 1/m$ is the shape parameter.

For the log-normal and the log-Weibull distributions the values $\log x_i$ are substituted for x_i.

For the Weibull distribution (μ known), the plotting positions are

$$\check{y}_i = \log \ln [-(1 - \check{P}_i)] \qquad (18$$

against which are plotted $\log x_i$ or $\log \log x_i$.

A straight line has now to be selected which yields the MAXNOR. A decision is much easier to do, if a preliminary line is fitted by eye, the deviations Δx_i or $\Delta \log x_i$ are computed and plotted against the plotting positions (16) - (18).

Graphs pertinent to normal distributions are presented in Figs. 5 and 6, to log-normal distributions in Fig. 7 and to Weibull distributions (μ known) in Fig. 8.

5.2 A computer program

Such a program, called the MAXNOR program, has been written by Göran Weibull for the three-parametric Weibull distribution. It proceeds like this: It starts by computing the three parameters of a curve passing through the data points 1,2,3. If the estimated location parameter x_u is larger than the least observation x_1 or smaller than a prescribed value, usually equal to zero, it goes on to the next point combination 1,2,4. Otherwise, the deviations are computed, squared and summed and the number-of-runs are counted. This procedure is systematically repeated for all combinations of the J data points. Only those combinations which yield MAXNOR are printed, as demonstrated in Fig. 4. The computing time for the 2,300 combinations was 22 sec by use of an IBM 360,M/75.

In view of the fact that there are, in general, several point combinations yielding MAXNOR, the sums of the squared deviations are presented. The parameters corresponding to MAXNOR and Least squares are accepted as preliminary estimates, in Fig. 4 being $x_u = 4.60$; $x_o = 0.647$, $m = 2.76$. An alternative program presents the Likelihood instead of the Least squares.

A decision between them has not yet been made.

A second program, called the M_{min} program, starts from the preceding estimates and seeks the Least squares, compatible with the given sample, either without any restriction or, restricted to an acceptable MAXNOR, in this particular case with $J = 25$, say, 9 or 10, which numbers correspond to the probabilites 7,6% and 15,4%, as demonstrated in Table 4.

6. Numerical examples

6.1 Normal distribution assumed

In a report by Argentiero & Tolson [5] a sample of 73 observations of "Red shifts of galaxies" was subjected to test for normality with the result that "the hypothesis that red shifts of galaxies in the Virgo Cluster are normally distributed cannot be rejected on the basis of available data". The decision was based on a χ^2 test with 9 classes, which yielded $\chi^2 = 5.29$. This is well below the 95-percent significance level, located at 15.5.

The MAXNOR criterion was applied to the sample $J/N = 9/73$, where the 9 observations were identical with the nine class limits used. The graph is presented in Fig. 5, where $\check{y}_i$ is equal to the medians of normal order statistics. A very acceptable MAXNOR = 5, corresponding to a $P_{nor} = 63.7\%$, was obtained.

It is, however, believed that almost any distribution function could be satisfactorily fitted to a sample of size 9, and that this size is too small for deciding whether an assumed distribution is acceptable or not.

In view of this fact, samples of various sizes were examined with the result presented below.

J/N	MAXNOR	P %	Accepted
9/73	5	63.7	Yes
18/73	7	16.6	Yes
37/73	17	30.9	Yes
49/73	17	1.5	No
73/73	19	$\ll 0.01$	No

The graph for the complete sample 73/73, plotted on a normal probability paper, is presented in Fig. 6. The fact that about 50 observations were needed for rejecting the hypothesis of normality may be taken as an indication that the departure from normality, even if unacceptable, is not very large.

6.2 Comparison between a log-normal and a two-parametric Weibull ($\mu = 0$) distribution

From an AFML Tech.Rep. by Sedlacek [6] three test series on tensile strength of Wesgo Al-995 Alumina specimens, each of size 10, were taken. After having checked that they statistically belonged to the same population, the MAXNOR criterion was applied to one of the series, to two of them pooled, and to all of them pooled. The graphs for $J/N = 30/30$ are presented in Fig. 7, assuming log-normal distribution, where the deviations of $\log x_i$ are plotted against $\check{P}_i$ on a normal probability paper, and in Fig. 8, assuming Weibull ($\mu = 0$) distribution, where the deviations of $\log x_i$ are plotted against $\check{y}_i$ from equ.(18). The results are presented below.

J/N	Log-normal			Weibull ($\mu = 0$)		
	MAX NOR	P %	Accept.	MAX NOR	P %	Accept.
10/10	7	91.0	Yes	7	91.0	Yes
20/20	12	82.0	Yes	12	82.0	Yes
30/30	9	1.0	No	14	35.6	Yes

The conclusion drawn from the preceding example, that a decision requires a not too small sample size, is confirmed by these results.

6.3 Comparison between three-parametric log-normal and Weibull distributions

In a paper by Serensen et al. [7], data from a very large rotating-bending fatigue test on aluminum alloy specimens are reported. The 463 fatigue life-times are grouped into 25 classes. The authors found that for this stress level ($\sigma = 17/\text{kg/mm}^2$), the fatigue-life distribution was not log-normal. By introducing a lower bound N_o, the quantity $\log(N - N_o)$ was supposed to be normally distributed.

Using the 25 upper class limits, the sample $J/N = 25/463$ was MAXNOR-tested for normality. The graph is presented in Fig.9, yielding a MAXNOR = 13, corresponding to a $P_{nor} = 58.1\%$ which result confirms the authors' assumption.

However, it has been proved by Dubey [8] that the normal distribution can be very closely approximated by the 3-parametric Weibull distribution with differences nowhere exceeding 0.78%, if the parameters: $m = 3.602$, $x_u = -3.243$, $x_o = 3.599$ are used.

In view of this fact a positive result of a MAXNOR test may be expected also under the assumption that $\log(N - N_o)$ is Weibull distributed. Two other assumptions were at the same time examined, giving the following three alternatives:

Weibull distribution is assumed for

1) $x = N$; 2) $x = \log N$; 3) $x = \log(N - N_o)$

The results are listed in Table 4.

The MAXNOR presented in Fig.4 is actually that of Alt.2. It is of interest to note that in Alt.3, the estimated shape parameter was $\hat{m} = 3.525$ which is very close to the Dubey value $m = 3.602$. It follows that all three assumptions may be accepted, with preference of Alt.2, due to the smallest M_{min}. Applying an unrestricted M_{min} program, results in substantial reductions in M_{min} but unacceptable MAXNORs. The restrictions MAXNOR ≥ 10 and ≥ 9, corresponding to 90% and 95% levels of significance, give preference also of Alt.2, which assumes that fatigue life is log-Weibull distributed. The same result has been observed on the basis of MAXNOR tests on a few quite large fatigue-test series. The small computing times of the M_{min}-programs are obtained by use of an IBM 360, M/75.

6.4 Three-parametric Weibull distribution assumed

From a paper by Oh & Finnie [9] data of two bending tests on glass plates were taken and analyzed. The graph of the first sample, $J/N = 38/38$, is presented in Fig.10. The assumption is clearly acceptable with a $P_{nor} = 37.1\%$.

The graph of the second sample, $J/N = 43/43$, is presented in Fig.11. The assumption is rejected on the basis of a $P_{nor} = 1.0\%$. A closer examination of this graph indicates that the sample is taken from a two-component population. Dividing the sample into two parts of sizes, $N_1 = 23$, $N_2 = 20$, yields very acceptable values of P_{nor} being 25% and 17.9%. The physical interpretation of the two components in this plain-bending test, is believed to be that they correspond to the two sides of the glass plates which, due to the manufacturing process, have different strength properties.

7. Conclusions

From the illustrative examples presented and from further experience it is concluded that the MAXNOR criterion provides a useful tool for deciding whether an assumed distribution function is acceptable or not, also which of two or more functions is the better one with regard to departure from the true function. However, no decisions can be made on the basis of small samples, which is believed to be a property of any criterion, and the power increases strongly with the number of observations available.

It is recommended to start the statistical analysis of any sample of test data by applying the MAXNOR test, because, if the result is negative, there is no sense in continuing the estimation procedure under the assumed conditions.

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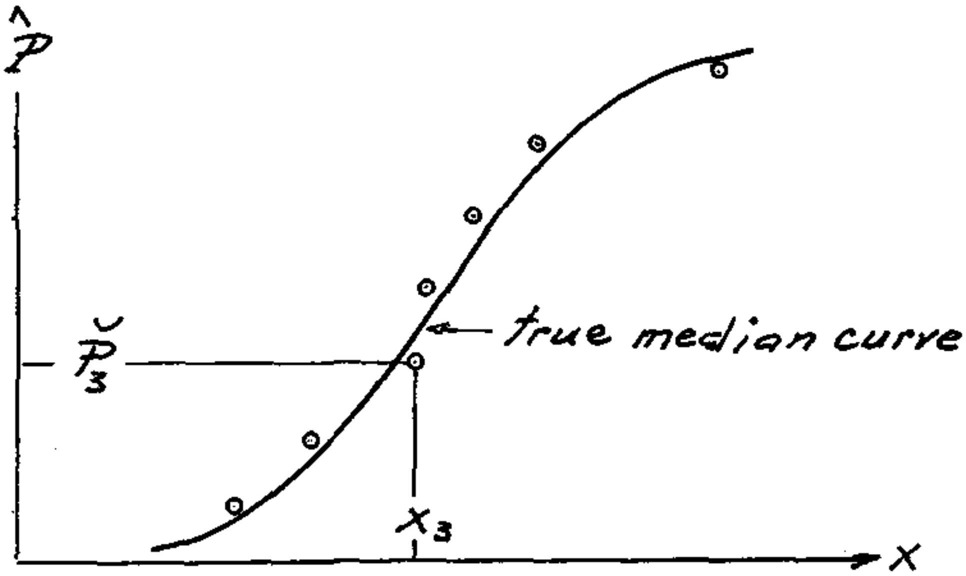

Fig. 1. Deviations from a median curve.

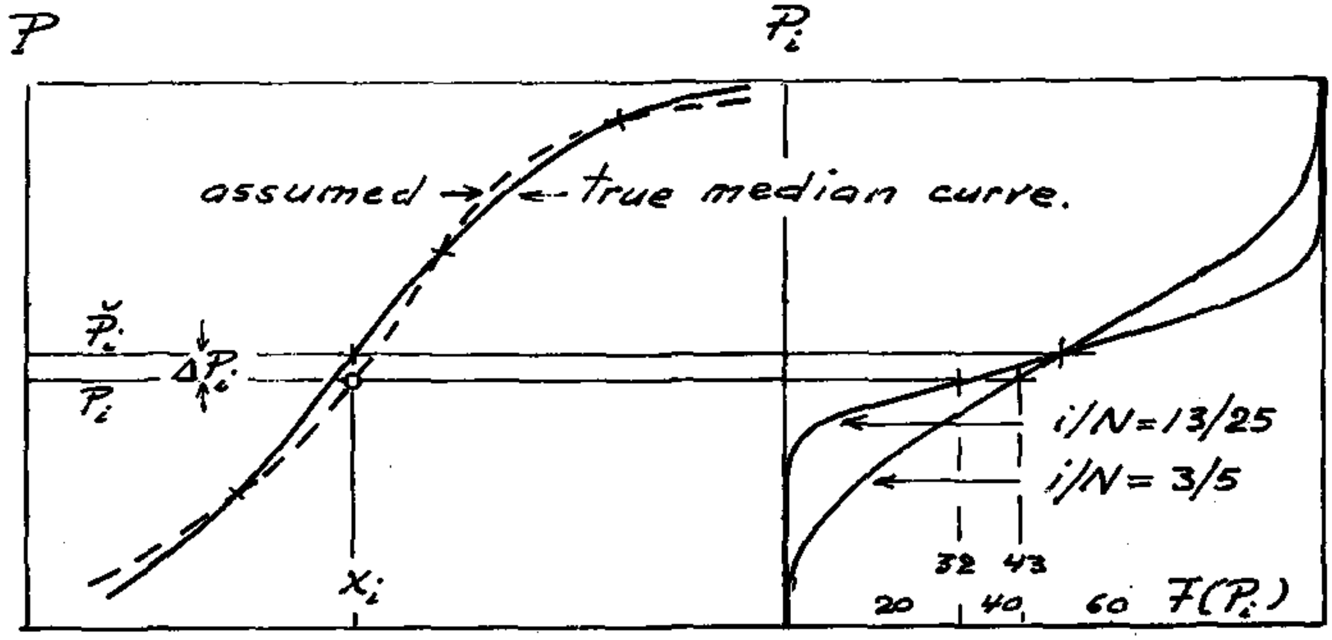

Fig. 2. An assumed and a true median curve.

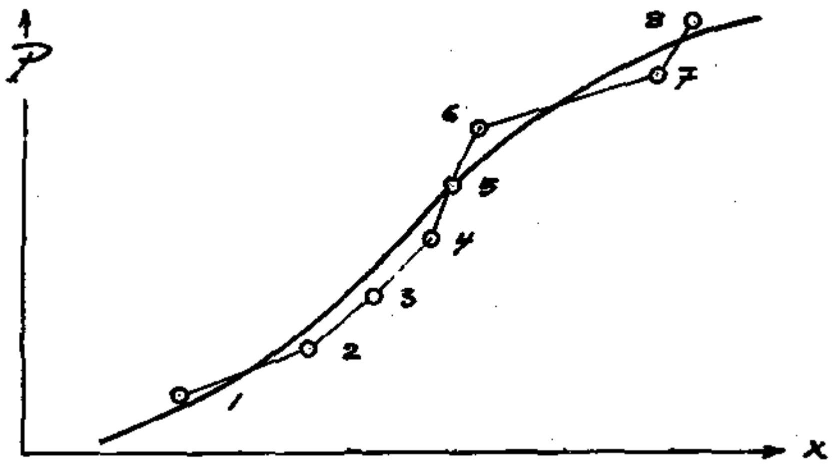

Fig. 3. A MAXNOR curve.

SERENSEN ET AL 2   x=log N   MAXNOR - program

SAMPLE SIZE = 463   J/N = 25/463

JMAX = 25   NONSA = 1   NORMIN = 4   MAXNOR = 13; P_{nor} = 58.1%

DNJ = 0.00010000   XUMIN = 0.0   LOP11 = 1   LOPMAX = 100000

JA	JB	JC	XU	XO	M	M MIN	NOR
3	5	18	0.460005E 01	0.648522E 00	0.276753E 01	0.979391E 03	13
3	5	24	0.460005E 01	0.649659E 00	0.276246E 01	0.107904E 04	13
3	5	25	0.460005E 01	0.648682E 00	0.276682E 01	0.992774E 03	13
3	6	18	0.460118E 01	0.646924E 00	0.275740E 01	0.957392E 03	13
3	6	24	0.460232E 01	0.646264E 00	0.274748E 01	0.100211E 04	13
3	18	25	0.460005E 01	0.648252E 00	0.276402E 01	0.968090E 03	13
5	9	17	0.457734E 01	0.676364E 00	0.290579E 01	0.140482E 04	13
5	9	18	0.457507E 01	0.678210E 00	0.292264E 01	0.135132E 04	13
5	17	18	0.455010E 01	0.707404E 00	0.307633E 01	0.186353E 04	13
5	17	23	0.455804E 01	0.698335E 00	0.302562E 01	0.171107E 04	13
5	17	24	0.459778E 01	0.652599E 00	0.277433E 01	0.113282E 04	13
5	17	25	0.459324E 01	0.657943E 00	0.280419E 01	0.118702E 04	13
5	18	23	0.456031E 01	0.695510E 00	0.301352E 01	0.163917E 04	13
6	9	17	0.455691E 01	0.697741E 00	0.299961E 01	0.151492E 04	13
6	9	18	0.455577E 01	0.698572E 00	0.300924E 01	0.147571E 04	13
6	17	18	0.453363E 01	0.723384E 00	0.313110E 01	0.179937E 04	13
6	17	23	0.454612E 01	0.709658E 00	0.306072E 01	0.164335E 04	13
6	17	24	0.459607E 01	0.654199E 00	0.277902E 01	0.113115E 04	13
6	17	25	0.458983E 01	0.661134E 00	0.281345E 01	0.118060E 04	13
6	18	23	0.454896E 01	0.706171E 00	0.304595E 01	0.156522E 04	13
6	18	25	0.460005E 01	0.648252E 00	0.276402E 01	0.968090E 03	13

ELAPSED TIME IS   21.92 SECONDS   TOTAL NOC = 2,300

REFERENCE TIME INITILIZED TO ZERO

Fig. 4. A MAXNOR PROGRAM

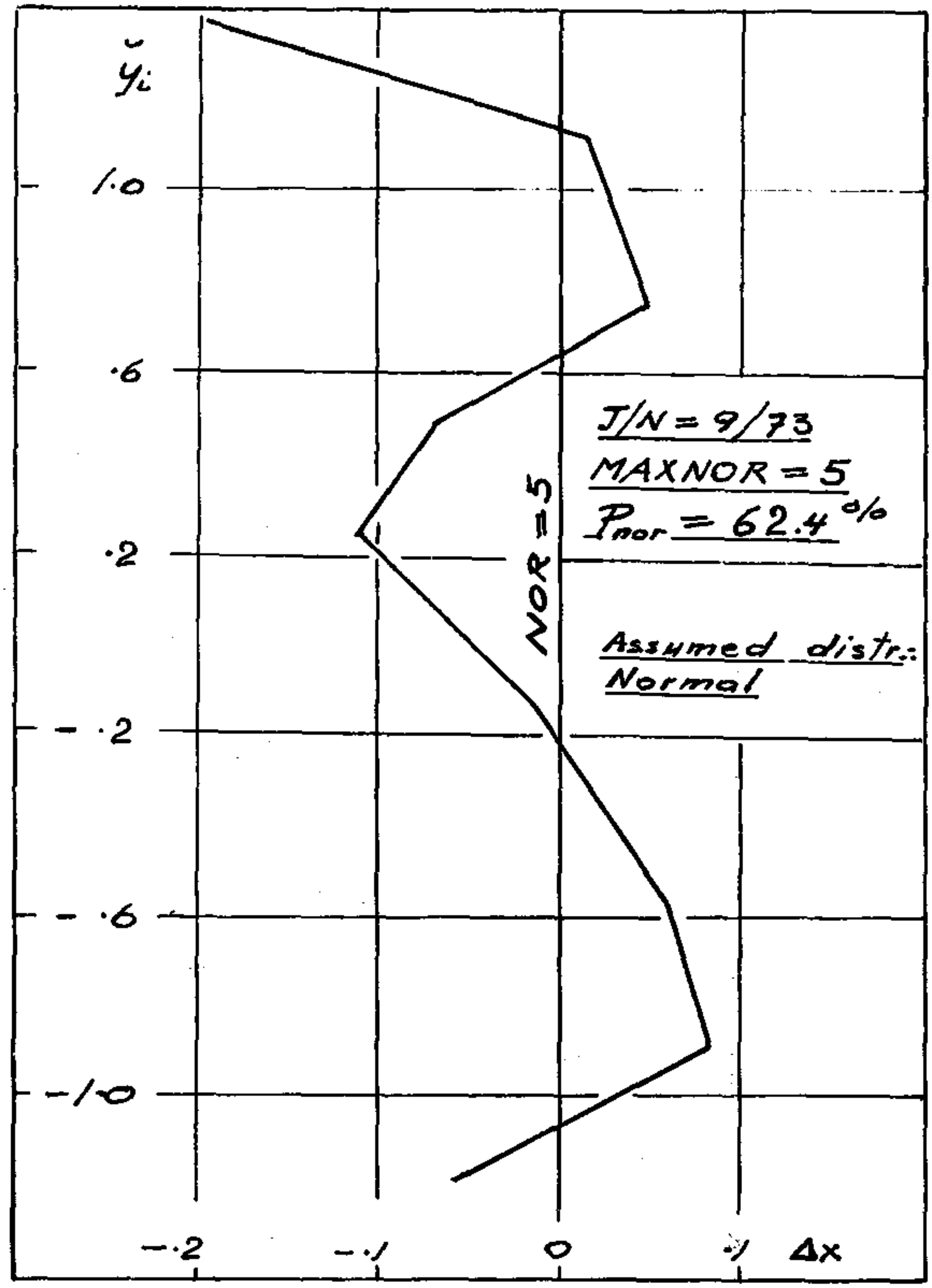

Fig. 5. Red Shifts in the Virgo Cluster.
9 observations

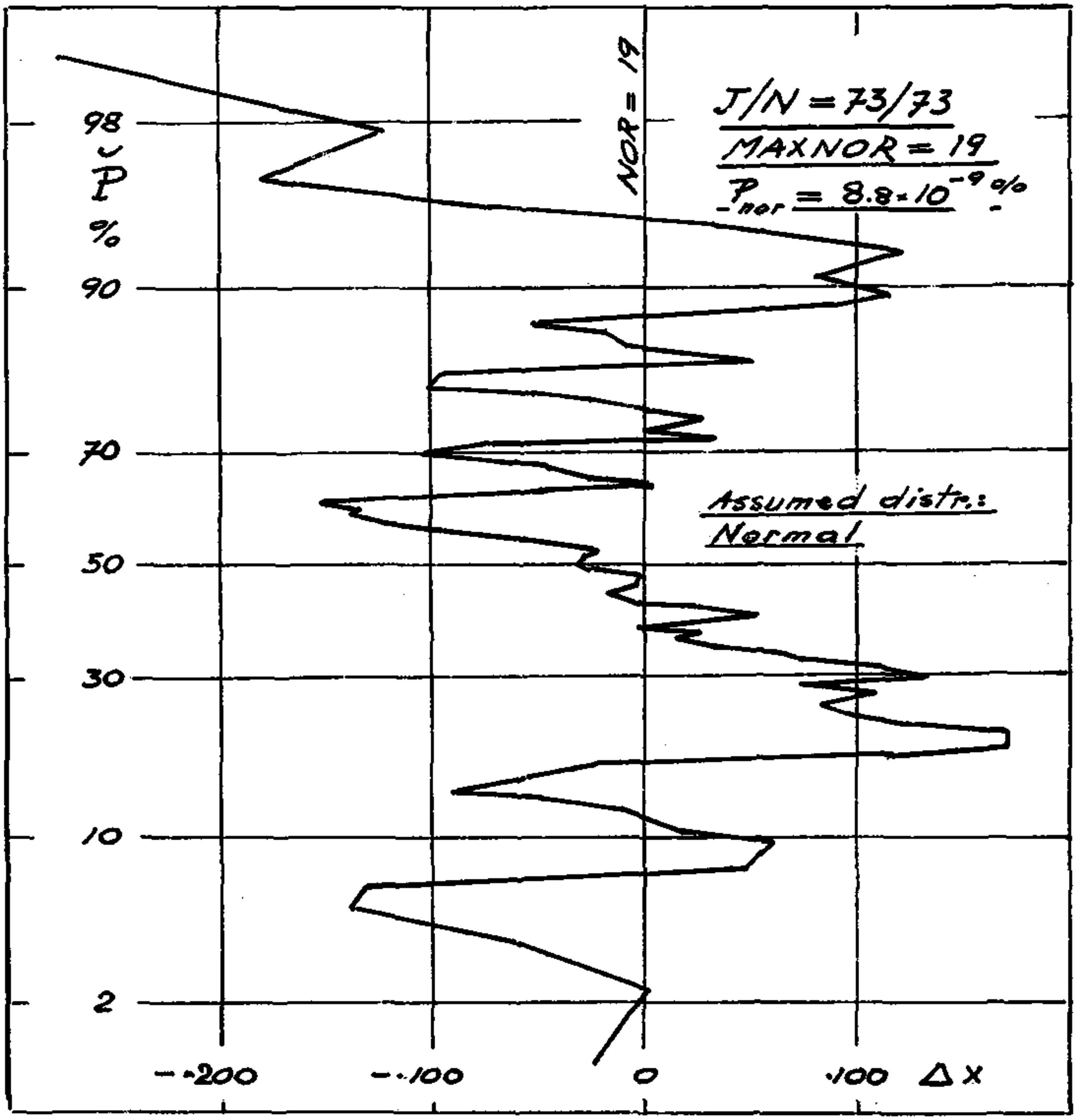

Fig. 6. Red Shifts in the Virgo Cluster. 50 observations

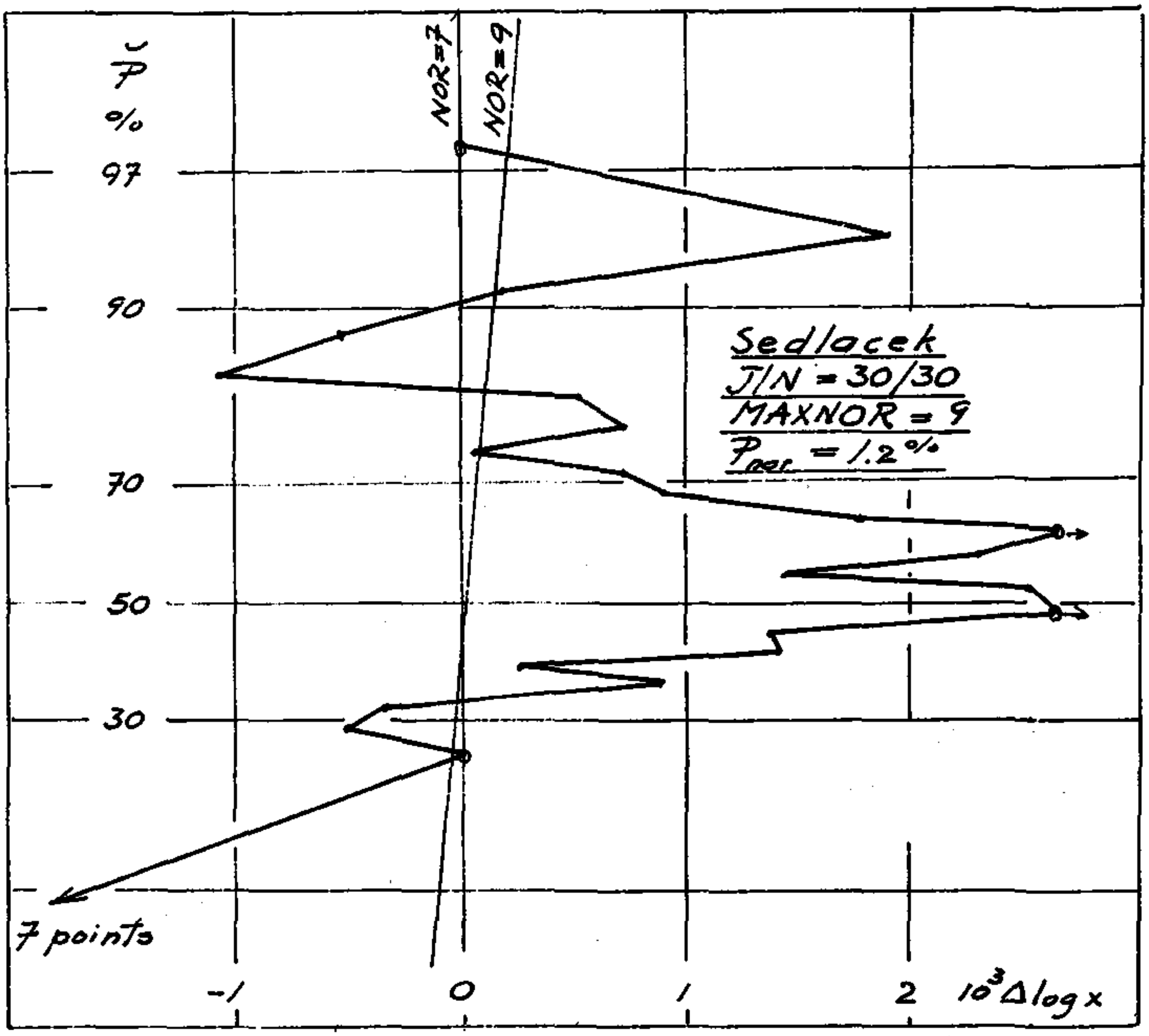

Fig. 7. Tensile strength of alumina specimens. Assumed Distribution: Log-normal.

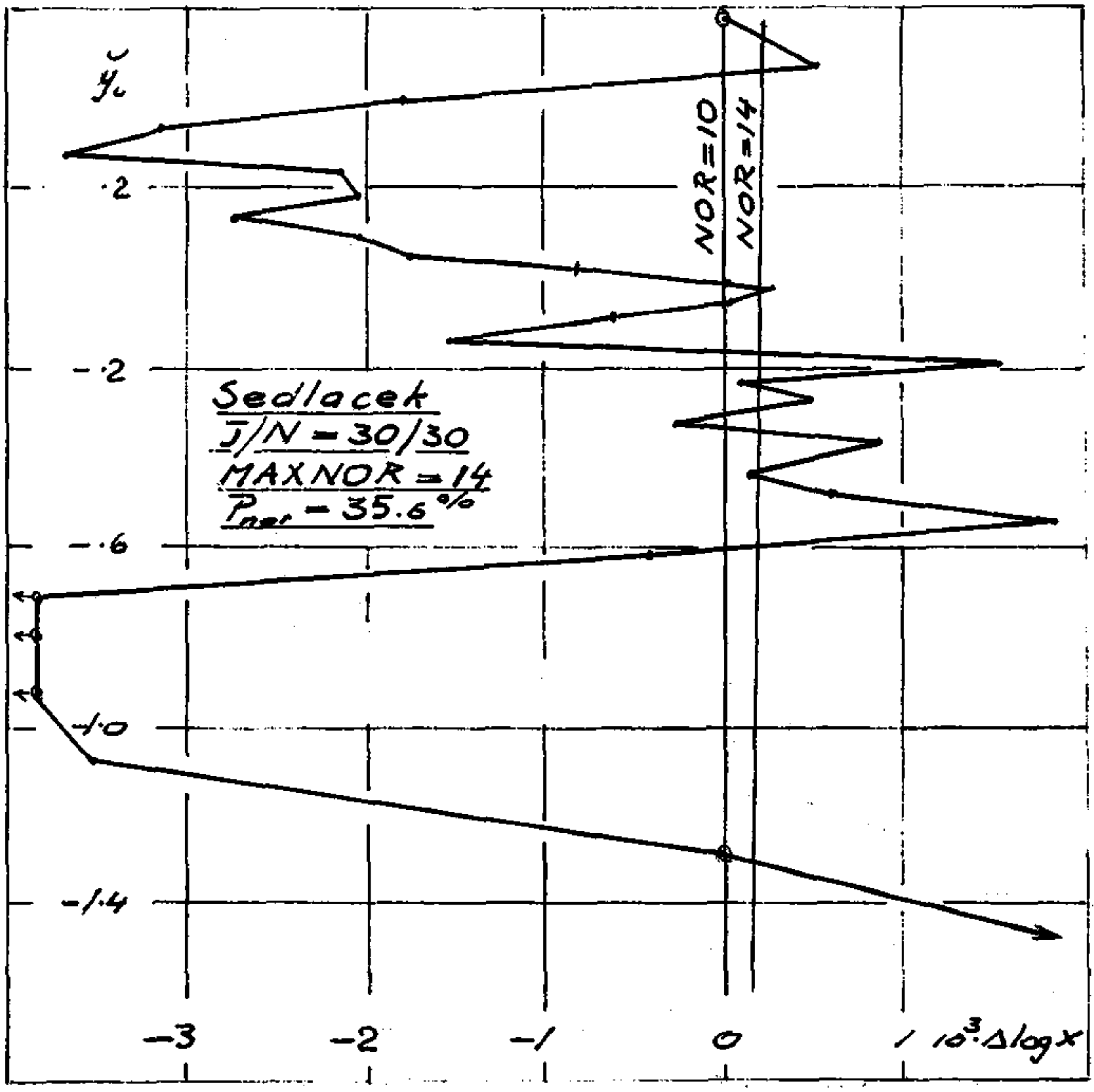

Fig. 8. Tensile strength of alumina specimens. Assumed Distribution: Weibull (X_u=0)

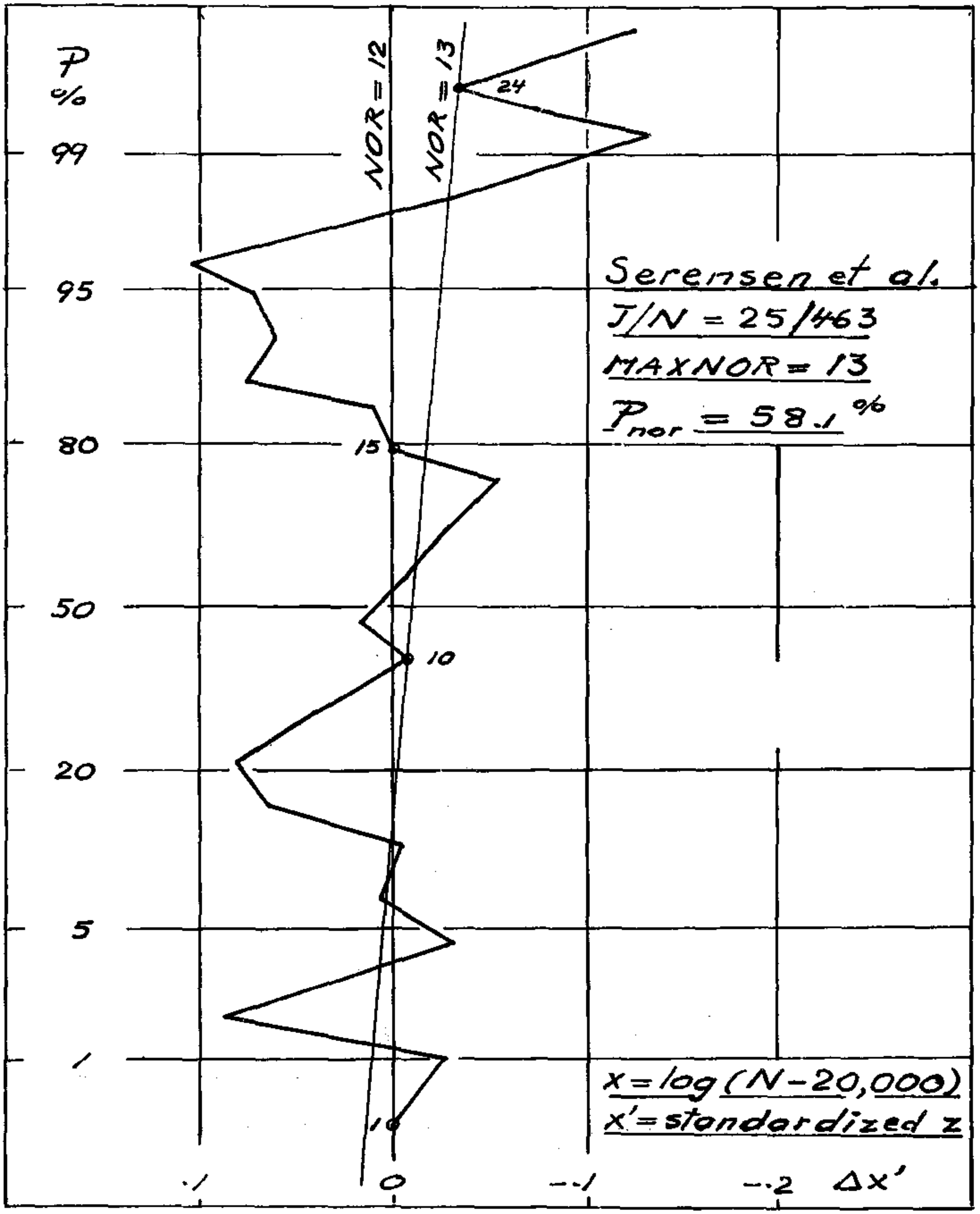

Fig. 9. Fatigue life of aluminum alloy specimens. Assumed Distribution: $Log(N-N_o)$-Normal

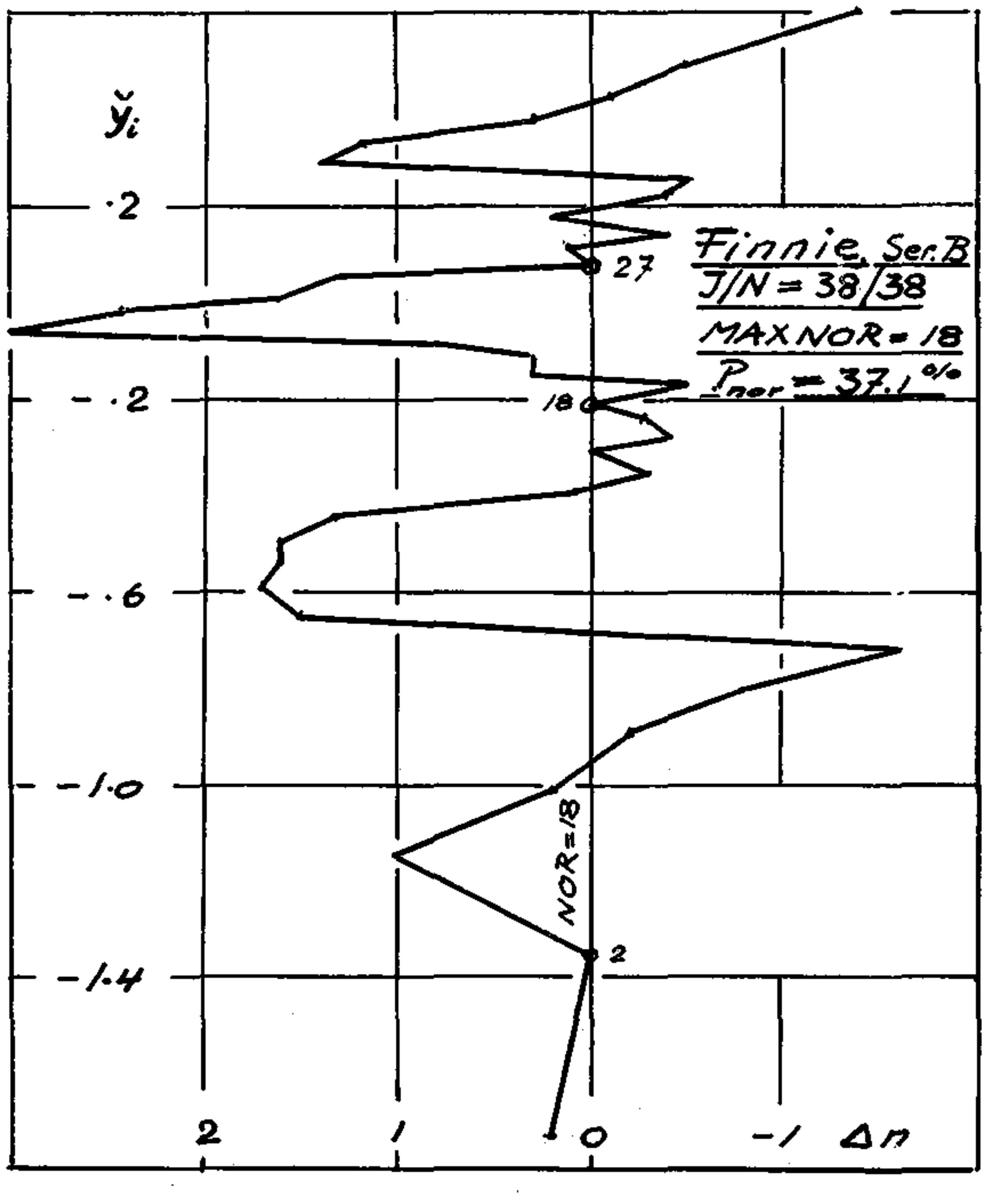

Fig. 10. Bending strength of glass plates. Assumed Distribution: 3-Param. Weibull

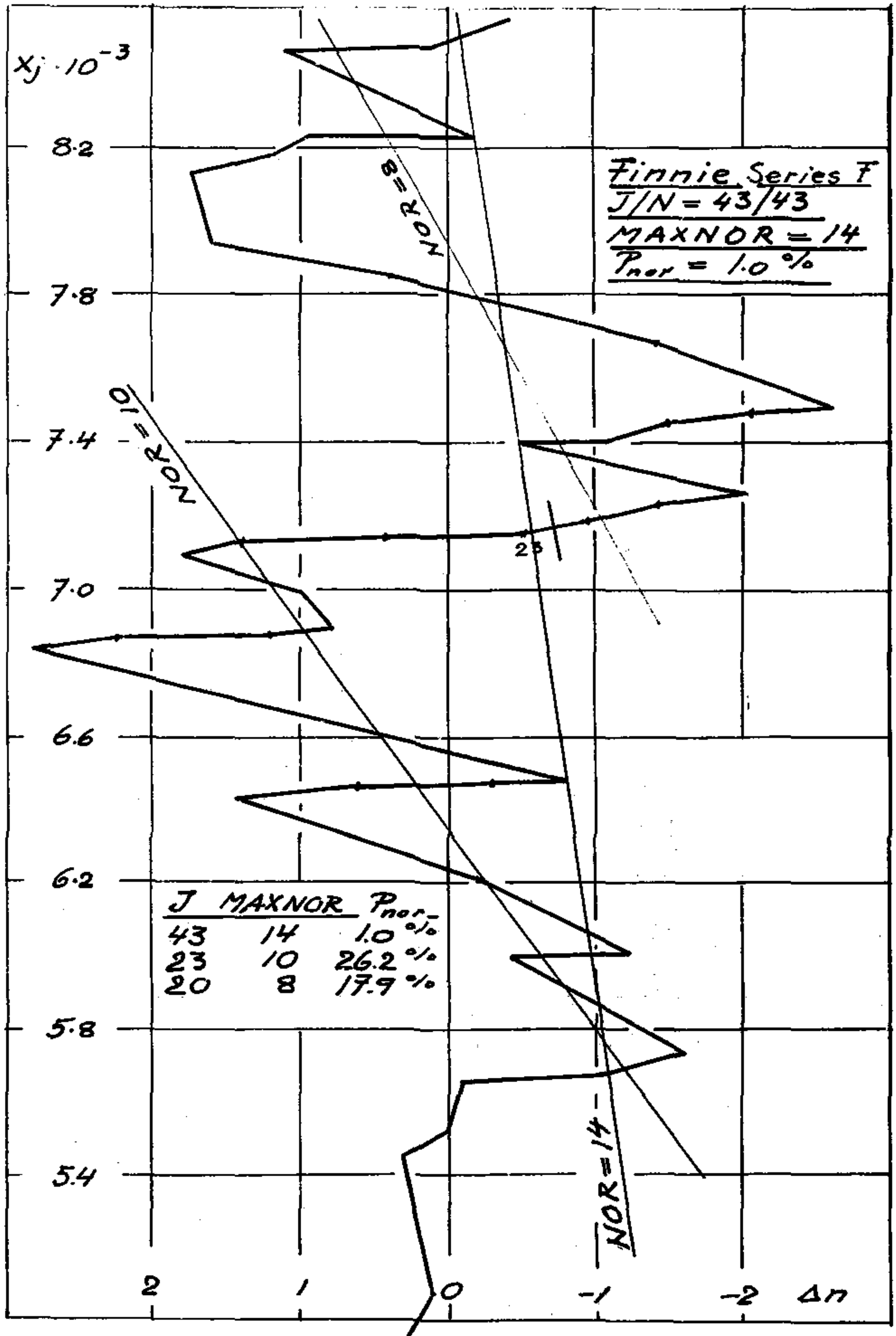

Fig.11. Bending strength of glass plates. Assumed Distribution: Weibull

Table 1. Number of arrangements having u runs for J = 2(1)5

J	u	Arrange-ments	Nr of arrang.
2	1	++ --	2.1
	2	+- -+	2.1
3	1	+++ ---	2.1
	2	++- --+ +-- -++	2.2
	3	+-+ -+-	2.1
4	1	++++ ----	2.1
	2	+++- ---+ ++-- --++ +--- -+++	2.3
	3	++-+ --+- +-++ -+-- +--+ -++-	2.3
	4	+-+- -+-+	2.1
5	1	+++++ -----	2.1
	2	++++- ----+ +++-- ---++ ++--- --+++ +---- -++++	2.4
	3	+++-+ ---+- ++-++ --+-- ++--+ --++- +-+++ -+--- +--++ -++-- +---+ -+++-	2.6
	4	++-+- --+-+ +-++- -+--+ +-+-- -+-++ +--+- -++-+	2.4
	5	+-+-+ -+-+-	2.1

Table 2. The probabilities F(u) for J = 6(1)50(10)100

u \ J	6	7	8	9	10	11	12	13	14	15	
1	3.12	1.56	0.78	0.39	0.20	0.10	0.05	0.02	0.01	0.01	
2	18.75	10.94	6.25	3.52	1.95	1.07	0.59	0.32	0.17	0.09	
3	50.00	34.38	22.66	14.45	8.98	5.47	3.27	1.93	1.12	0.65	1 %
4	81.25	65.62	50.00	36.33	25.39	17.19	11.33	7.30	4.61	2.87	5 %
5	96.88	89.06	77.34	63.67	50.00	37.70	27.44	19.38	13.34	8.98	10 %
6	100.00	98.44	93.75	85.55	74.61	62.30	50.00	38.72	29.05	21.20	
7	-	100.00	99.22	96.48	91.02	82.81	72.56	61.28	50.00	39.53	50 %
8	-	-	100.00	99.61	98.05	94.53	88.67	80.62	70.95	60.47	
9	-	-	-	100.00	99.80	98.93	96.73	92.70	86.66	78.80	90 %
10	-	-	-	-	100.00	99.90	99.41	98.07	95.39	91.02	
11	-	-	-	-	-	100.00	99.95	99.68	98.88	97.13	

u \ J	16	17	18	19	20	21	22	23	24	25	
2	0.05	0.03	0.01	0.01	-	-	-	-	-	-	
3	0.37	0.21	0.12	0.07	0.04	0.02	0.01	0.01	-	-	
4	1.76	1.06	0.64	0.38	0.22	0.13	0.07	0.04	0.02	0.01	
5	5.92	3.84	2.45	1.54	0.96	0.59	0.36	0.22	0.13	0.08	
6	15.09	10.51	7.17	4.81	3.18	2.07	1.33	0.85	0.53	0.33	
7	30.36	22.72	16.62	11.89	8.35	5.77	3.92	2.62	1.73	1.13	1 %
8	50.00	40.18	31.45	24.03	17.96	13.16	9.46	6.69	4.66	3.20	
9	69.64	59.82	50.00	40.73	32.38	24.22	19.17	14.31	10.50	7.58	5 %
10	84.91	77.28	68.55	59.27	50.00	41.19	33.18	26.17	20.24	15.37	10 %
11	94.08	89.49	83.38	75.97	67.62	58.81	50.00	41.59	33.89	27.06	
12	98.24	96.16	92.83	88.11	82.04	75.78	66.82	58.41	50.00	41.94	
13	99.63	98.94	97.55	95.19	91.65	86.84	80.83	73.83	66.11	58.06	50 %
14	99.95	99.79	99.36	98.46	99.04	94.23	90.54	85.69	79.76	72.94	
15	99.99	99.97	99.88	99.62	99.78	97.93	96.08	93.31	89.50	84.63	90 %
16	100.00	99.99	99.99	99.93	99.96	99.41	98.67	97.38	95.34	92.42	

Table 2. (Continued)

u \ J	26	27	28	29	30	31	32	33	34	35	
5	0.04	0.03	0.02	0.01	0.01	-	-	-	-	-	
6	0.20	0.12	0.08	0.04	0.03	0.02	0.01	0.01	-	-	
7	0.73	0.47	0.30	0.19	0.12	0.07	0.04	0.03	0.02	0.01	
8	2.16	1.45	0.96	0.63	0.41	0.26	0.17	0.10	0.07	0.04	
9	5.39	3.78	2.61	1.78	1.21	0.81	0.53	0.35	0.23	0.15	
10	11.48	8.43	6.10	4.36	3.07	2.14	1.47	1.00	0.68	0.45	1%
11	21.22	16.35	12.39	9.25	6.80	4.94	3.54	2.50	1.75	1.22	
12	34.50	27.86	22.10	17.25	13.25	10.02	7.48	5.50	4.01	2.88	5%
13	50.00	42.25	35.06	28.58	22.92	18.08	14.05	10.77	8.14	6.07	10%
14	65.50	57.75	50.00	42.53	35.55	29.23	23.66	18.85	14.82	11.47	
15	78.78	72.14	64.94	57.47	50.00	42.78	36.00	29.84	24.34	19.58	
16	88.52	83.65	77.90	71.42	64.45	57.22	50.00	43.00	36.42	30.38	
17	94.61	91.57	87.61	82.75	77.08	70.77	64.00	57.00	50.00	43.21	50%
18	97.84	96.22	93.90	90.75	86.75	81.92	76.34	70.16	63.58	56.79	
19	99.27	98.55	97.39	95.64	93.20	89.98	85.95	81.15	75.66	69.62	
20	99.80	99.53	99.04	98.32	96.93	95.06	92.52	89.23	85.18	80.42	
21	99.96	99.88	99.70	99.37	98.79	97.86	96.46	94.50	91.86	88.53	90%
22	99.99	99.97	99.92	99.81	99.59	99.19	98.53	97.50	95.99	93.93	

u \ J	36	37	38	39	40	41	42	43	44	45	
8	0.03	0.02	0.01	0.01	-	-	-	-	-	-	
9	0.09	0.06	0.04	0.02	0.01	0.01	0.01	-	-	-	
10	0.30	0.20	0.13	0.08	0.05	0.03	0.02	0.01	0.01	0.01	
11	0.83	0.57	0.38	0.25	0.17	0.11	0.07	0.05	0.03	0.02	
12	2.05	1.44	1.00	0.69	0.47	0.32	0.22	0.14	0.10	0.06	
13	4.48	3.26	2.35	1.68	1.19	0.83	0.58	0.40	0.27	0.18	
14	8.77	6.62	4.94	3.65	2.66	1.92	1.38	0.98	0.69	0.48	1%
15	15.52	12.15	9.39	7.16	5.41	4.03	2.98	2.18	1.58	1.13	
16	24.98	20.26	16.20	12.80	9.98	7.69	5.86	4.42	3.30	2.44	
17	36.80	30.88	25.57	20.88	16.84	13.41	10.55	8.21	6.32	4.81	5%
18	50.00	43.40	37.14	31.36	26.12	21.48	17.44	14.00	11.10	8.71	10%
19	63.20	56.60	50.00	43.58	37.46	31.79	26.64	22.04	18.02	14.56	
20	75.02	69.12	62.86	56.42	50.00	43.73	37.76	32.20	27.12	22.57	
21	84.48	79.74	74.43	68.64	62.54	56.27	50.00	43.88	38.04	32.58	
22	91.23	87.85	83.80	79.12	73.88	68.21	62.24	56.12	50.00	44.02	50%
23	95.52	93.38	90.61	87.20	83.16	78.52	73.36	67.80	61.96	55.98	
24	97.95	96.74	95.06	92.84	90.02	86.59	82.56	77.96	72.88	67.42	
25	99.17	98.56	97.65	96.35	94.59	92.31	89.45	86.00	81.98	77.43	
26	99.70	99.43	99.00	98.32	97.34	95.97	94.14	91.79	88.90	85.44	90%
27	99.91	99.80	99.62	99.31	98.81	98.08	97.02	95.58	93.68	91.29	
28	99.97	99.94	99.87	99.75	99.53	99.17	98.62	97.82	96.70	95.19	

Table 2. (Continued)

u \ J	46	47	48	49	50	60	70	80	90	100	
11	0.01	0.01	-	-	-	-	-	-	-	-	
12	0.04	0.03	0.02	0.01	0.01	-	-	-	-	-	
13	0.12	0.08	0.05	0.04	0.02	-	-	-	-	-	
14	0.33	0.23	0.15	0.10	0.07	-	-	-	-	-	
15	0.80	0.57	0.40	0.28	0.19	-	-	-	-	-	
16	1.78	1.29	0.93	0.66	0.47	0.01	-	-	-	-	
17	3.62	2.70	2.00	1.47	1.06	0.03	-	-	-	-	
18	6.76	5.19	3.95	2.97	2.22	0.08	-	-	-	-	
19	11.64	9.20	7.20	5.57	4.27	0.19	-	-	-	-	
20	18.56	15.10	12.15	9.67	7.62	0.43	0.01	-	-	-	
21	27.58	23.06	19.02	15.62	12.64	0.92	0.03	-	-	-	
22	38.30	32.94	28.00	23.54	19.58	1.82	0.07	-	-	-	
23	50.00	44.14	38.54	33.27	28.40	3.37	0.17	-	-	-	
24	61.70	55.86	50.00	44.28	38.77	5.88	0.38	0.01	-	-	
25	72.42	67.06	61.46	55.72	50.00	9.63	0.77	0.03	-	-	
26	81.44	76.94	72.00	66.73	61.23	14.88	1.47	0.07	-	-	
27	88.36	84.90	80.91	76.46	71.60	21.75	2.66	0.16	-	-	
28	93.24	90.80	87.85	84.38	80.42	30.15	4.56	0.33	0.01	-	
29	96.38	94.81	92.80	90.33	87.36	39.74	7.40	0.64	0.04	-	
30	98.22	97.30	96.05	94.43	92.38	50.00	11.42	1.19	0.07	-	
31	-	-	-	-	-	-	16.78	2.11	0.15	-	
32	-	-	-	-	-	-	23.52	3.56	0.28	-	
33	-	-	-	-	-	-	31.52	5.73	0.54	0.01	
34	-	-	-	-	-	-	40.50	8.83	0.97	0.04	
35	-	-	-	-	-	-	50.00	13.02	1.68	0.12	
36	-	-	-	-	-	-	-	18.41	2.80	0.23	
37	-	-	-	-	-	-	-	25.00	4.47	0.43	
38	-	-	-	-	-	-	-	32.65	6.88	0.77	1%
39	-	-	-	-	-	-	-	41.11	10.16	1.33	
40	-	-	-	-	-	-	-	50.00	14.46	2.19	
41	-	-	-	-	-	-	-	-	19.84	3.49	5%
42	-	-	-	-	-	-	-	-	26.26	5.37	
43	-	-	-	-	-	-	-	-	33.59	7.95	10%
44	-	-	-	-	-	-	-	-	41.61	11.38	
45	-	-	-	-	-	-	-	-	50.00	15.74	
46	-	-	-	-	-	-	-	-	-	21.08	
47	-	-	-	-	-	-	-	-	-	27.34	
48	-	-	-	-	-	-	-	-	-	34.39	
49	-	-	-	-	-	-	-	-	-	42.04	50%
50	-	-	-	-	-	-	-	-	-	50.00	

Table 3. Values of $F(u)$ in per cent for various p_i

$N = 5$

u \ p_i	0.50	0.60	0.75	0.90	1.00
1	6.25	8.80	23.83	59.05	100.00
2	31.25	33.76	47.27	73.81	-
3	68.75	71.20	82.42	95.95	-
4	93.95	94.24	96.48	99.19	-
5	100.00	100.00	100.00	100.00	-
$\Sigma u.f(u) =$	3.00	2.92	2.50	1.72	1.00

$N = 10$

u \ p_i	0.50	0.60	0.75	0.90	1.00
1	0.20	0.62	5.63	34.87	100.00
2	1.95	4.15	11.26	43.58	-
3	8.98	18.28	33.79	78.45	-
4	25.39	35.25	50.38	85.81	-
5	50.00	69.72	75.27	96.84	-
$\Sigma u.f(u) =$	5.5	-	-	2.4	1.0

Table 4. Rotating-bending fatigue tests on Al-alloy specimens

Data from Serensen, Kogaev, Stepnov and Glatsintov

Series	MAXNOR-program					M_{min} - program unrestricted			
	MAX NOR	P %	M_{min}	NOC	Comp time sec.	M_{min}	NOR	P %	Comp time sec.
1	11	27.1	3,901	7	9.33	1,924	5	0.1	2.7
2	13	58.1	957	21	21.92	553	7	1.1	2.4
3	13	58.1	1,063	14	20.77	593	5	0.1	2.8

Series	M_{min} - program restricted MAXNOR 10				M_{min} - program restricted MAXNOR 9			
	M_{min}	NOR	P %	Comp time sec.	M_{min}	NOR	P %	Comp time sec.
1	3,901	11	27.1	2.1	2,355	9	7.6	3.5
2	751	11	27.1	3.2	609	9	7.6	2.4
3	1,018	11	27.1	2.5	659	9	7.6	2.7

Series 1: $x = N$; Series 2: $x = \log N$; Series 3: $x = \log(N - 2.10^4)$

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13. ABSTRACT

The usual way to solve the fundamental problem of deciding whether an assumed distribution function is acceptable or not, consists - if at all done - in estimating the parameters and checking the attained goodness of fit by some accepted criterion, in most cases the Chi-square test. In this way, the decision depends not only on the assumed function, so it may happen that an acceptable function may be rejected on the basis of results from poor estimating or fitting procedures.

The purpose of this research was to find a criterion which eliminates such fatalities and depends entirely on the assumed function alone. Such a criterion, based on the "number-of-runs" has been proposed. The properties of this statistic and its usefulness as a measure of departure from the true distribution have been demonstrated. The concept "maximum number of runs" (MAXNOR) of a given sample and methods for its ascertaining have been introduced. Its use as a criterion for deciding whether the assumed function is acceptable or not has been studied by applying it to data from tests on strength of brittle materials, fatigue life of aluminum alloys, etc.

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Statistics New criterion for acceptance of assumed distribution function Application to test data - strength of brittle materials, fatigue of aluminum alloys etc.						