5. A generalized distribution function

A variate Z, defined by its density function

$$f(z,r,m) = \frac{r}{(r/m)!} z^{r-1+0} e^{-z^m} \qquad \begin{matrix}(r > 0)\\(m > 0)\end{matrix} \tag{67}$$

presents a distribution which by proper specification of its two shape parameters r and m involves as special cases the following distributions: the Exponential, the Gamma, the Pearson Type III, the Chi-square, the Sampling distribution of the standard deviation from a normal population, the Rayleigh, the Weibull, and some more distributions of importance.

A location parameter μ and a scale parameter β may be introduced by putting

$$z = (x-\mu)/\beta \tag{68}$$

Z will be called the standardized variate X ($\mu = 0$, $\beta = 1$). Its cumulative distribution function will be denoted $F(z,r,m)$.

The following special distributions will be examined:

Parameter $m = 1$

a) Exponential: $r = 1$; $f(z,1,1) = z^{+0} \cdot e^{-z}$

b) Gamma: r arbitrary; $f(z,r,1) = z^{r-1+0} \cdot e^{-z}/(r-1)!$

c) Pearson Type III = the preceding by introducing (68)

d) Chi-square: $r = n/2$; $f(\frac{z}{2}, \frac{n}{2}, 1) = k \cdot z^{(n/2)-1+0} \cdot e^{-z/2}$

Parameter $m = 2$

e) Sampling distribution of the standard deviation from a normal population: r = arbitrary positive integer,

$$f(\frac{z^2}{a}, n, 2) = k \cdot x^{n-1+0} \cdot e^{-x^2/2\sigma^2}$$

f) Distributions such as that of the deviations of a shot from the target, that of the velocity of a molecule, etc.

g) Rayleigh: $r = m$. This distribution is a special case of the following.

Parameter m arbitrary:

h) Weibull: $r = m$; $f(x,m,m) = m \cdot z^{m-1+0} \cdot e^{-z^m}$

The general density function will now be expanded in a power series and transformed into an integral series. The integral series of the Exponential, the Gamma, and the Weibull density functions will also be presented, the latter in two different types.

6. Expansion in power series and integral series

The power series of the generalized density function is:

$$f(z,r,m) = \frac{r}{(r/m)!}\left(z^{r-1}/0! - z^{m+r-1}/1! + z^{2m+r-1}/2! - \dots\right)z^{+0} \qquad (69)$$

and by equ.(63) the corresponding integral series

$$f(z,r,m) = \frac{r}{(r/m)!}\,\Sigma(-i)^2 \cdot \frac{(i.m+r-1)!}{i!} \cdot j^{-(i.m+r)}[0] \quad (i=0,1,2..) \qquad (70)$$

The integral series of the cumulative distribution function $F(z,r,m)$ is obtained by multiplying each term by j^{-1}

For the special cases we have

The Exponential function

$$f(z,1,1) = (j^{-1} - j^{-2} + j^{-3} - \dots)[0] \qquad (71)$$

The Gamma function

$$f(z,r,1) = \left(\frac{1}{0!}\,j^{-r} - \frac{r}{1!}\,j^{-(r+1)} + \frac{r(r+1)}{2!}\,j^{-(r+2)} - \frac{r(r+1)(r+2)}{3!}\,j^{-(r+3)} \qquad (72)$$

For $r = 1$ this series reduces to the Exponential.

The Weibull function

$$f(z,m,m) = m\left(\frac{(m-1)!}{0!}\,j^{-m} - \frac{(2m-1)!}{1!}\,j^{-2m} + \frac{(3m-1)!}{2!}\,j^{-3m} - \dots\right)[0] \qquad (73)$$

This series reduces to the Exponential for $m = 1$ and to the Rayleigh for $m = 2$.

The indicated series are suitable for small values of z, but many terms are required for large values of z. In some cases it will be more convenient to use series of the form:

$$f(z) = \Sigma k_i (z - b_i)^{m_i + 0} = \Sigma k_i \cdot m_i ! \, j^{-(m_i+1)} [b_i] \qquad (74)$$

A comparison between this series and the series (73) has been performed for $m = 2$.

By (73) we have

$$F(z,2,2) = \left(\frac{z^2}{1!} - \frac{z^4}{2!} + \frac{z^6}{3!} - \frac{z^8}{4!} + \frac{z^{10}}{5!} \right) \qquad (75)$$

Values of this function have been computed for $z = 0(0.1)1.5$ alternatively with one, three, and five terms and are presented in Table II. If an error of about 0.1% is accepted, one term is sufficient up to $F = 4\%$, three terms up to $F = 30\%$, and five terms up to $F = 63\%$.

The coefficients of the series (74) have been determined in the following way: By plotting the exact values of $\log F(z,2,2)$ against $\log z$, it was found that a good approximation results by drawing a straight line through the points corresponding to $z = 0.2$ and 0.4.

The first term of the series thus becomes for $b_1 = 0$

$$F(z,2,2) = 0.8540 \, z^{1.9143} \qquad (76)$$

which may be used up to $F = 22\%$. The values are exact for $z = 0$, 0.2, and 0.4.

The second term is applied to the differences between the exact curve and the first term by choosing $b_2 = 0.4$ to give exact values for $x = 0.4$, 0.6, and 0.8. In the same way, the third term is determined with $b_3 = 0.8$ in such a way that exact values are obtained for $x = 0.8$, 1.0, and 1.3.

The final series becomes

$$F(z,2,2) = 0.8540 \, z^{1.9143} - 0.6128(z - 0.4)^{2.1648} - 0.4024(z-0.8)^{1.9002} \qquad (77)$$

This formula gives exact values at the points $z = 0$, 0.2, 0.4, 0.6, 0.8, 1.0 and 1.3.

The values of this series are presented in Table II. It is acceptable up to $F = 86\%$.

The density function, obtained by differentiation, is then transformed into an integral series, which becomes;

$$f(z,2,2) = 1.5748\ j^{-1.9143}[0] - 1.7031\ j^{-2.1648}[0.4] - 0.7354\ j^{-1.9002}[0.8] \quad (78)$$

7. Composition of independent variates

By use of the integral series the composition of variates is easily performed, as will now be demonstrated.

The sum of two discrete variates $X = \Sigma a_i[x_i]$ and $Y = \Sigma b_j[y_j]$ is performed according to the following scheme

$$X + Y = \Sigma \Sigma a_i \cdot b_j[x_i + y_j] \quad (79)$$

This scheme applies also if a_i and b_j are equal to discrete integral masses of the type j^{-m}.

The easiest way of multiplying two variates consists in adding the logarithms according the scheme

$$\log(X \cdot Y) = \log X + \log Y \quad (80)$$

where $\log X$ is determined by equ.(24).

The procedure (79) will be demonstrated by means of a simple example:

Let the two variates be

$$Z_1 = (z - b)^{m+0}[z] = m!\, j^{-(m+1)}[b] \quad (81)$$

and

$$Z_2 = (z - c)^{n+0}[z] = n!\, j^{-(n+1)}[c] \quad (82)$$

Hence by (79)

$$Z_1 + Z_2 = m!\, n!\, j^{-(m+n+2)}[b+c] \quad (83)$$

and by (63)

$$f(z) = m!\,n!\,(z-b-c)^{m+n+1+0}/(m+n+1)! \tag{84}$$

This random addition can be performed also by means of the classical method of convolution in the following way:

The density function $f(z)$ of a sum of two independent variates with the density functions $f_1(x)$ and $f_2(y)$ is

$$f(z) = \int_{-\infty}^{\infty} f_2(z-x)\,.\,f_1(x)dx \tag{85}$$

Since, in this particular case, the density functions are equal to zero at and below $x = b$ and $y = c$, we have

$$f(z) = \int_{b}^{z-c} (z-c-x)^n\,.\,(x-b)^m dx \tag{86}$$

Introducing the variable

$$t = (x-b)/(z-b-c) \quad \text{or} \quad x = (z-b-c)\,t + b$$

$$dx = (z-b-c)\,dt \quad \text{and the limits} \quad t_1 = 1 \quad t_2 = 0$$

we have

$$f(z) = \int_{0}^{1} (z-b-c-(z-b-c)t)^m\,.\,(z-b-c)^n\,.\,(z-b-c)\,dt$$

$$f(z) = (z-b-c)^{m+n+1}\,.\,\int_{0}^{1} (1-t)^m\,.\,t^n dt$$

The integral is known as the complete Beta function and its value is equal to $m!\,n!/(m+n+1/!$ for $(m,n > -1)$

Hence

$$f(z) = m!\,n!\,(z-b-c)^{m+n+1+0}/(m+n+1)!$$

which is identical with equ.(84).

7.1 Addition of two Gamma variates

Consider two Gamma variates $Z(r)$ and $Z(s)$ with the density functions

$$f(z,r,1) = j^{-r} - r\,j^{-(r+1)}/1! + r(r+1)\,j^{-(r+2)}/2! - \ldots)[0] \tag{87}$$

and

$$f(z,s,1) = j^{-s} - s\,j^{-(s+1)}/1! + s(s+1)\,j^{-(s+2)}/2! - \ldots)\,[0] \qquad (88)$$

Hence by (79), the density function of their sum is

$$f_2(z) = j^{-(r+s)} - (r+s)\,j^{-(r+s+1)}/1! + (r+s)(r+s+1)\,j^{-(r+s+2)}/2!)\,[0] \qquad (89)$$

Comparing (89) and (88) or (87) it is readily found that

$$f_2(z) = f(z,(r+s),1)$$

Hence

$$Z(r) + Z(s) = Z(r+s)$$

that is, the sum of two independent Gamma variates with the parameters r and s is another Gamma variate with the parameter $(r+s)$.

Considering that the exponential variate is a Gamma variate with the parameter $r=1$, it follows that the sum of r independent exponential variates is a Gamma variate with the parameter r.

7.2 Addition of two Weibull variates

Consider two Weibull variates $Z(m)$ and $Z(n)$ with the density functions

$$f(z,m,m) = m\left(\frac{(m-1)!}{0!}\,j^{-m} - \frac{(2m-1)!}{1!}\,j^{-2m}\ \ldots\right)\,[0]$$

$$f(z,n,n) = n\left(\frac{(n-1)!}{1!}\,j^{-n} - \frac{(2n+1)!}{1!}\,j^{-2n}\ \ldots\right)\,[0]$$

Hence, by (79) the density function of their sum is

$$f_2(z,m,n) = m.n\left(\frac{(m-1)!\,(n-1)!}{0!\,0!}\,j^{-(m+n)} - \frac{(m-1)!\,(2n+1)!}{0!\,1!}\,j^{-m-2n} - \frac{(n-1)!\,(2m-1)!}{0!\,1!}\,j^{-n-2m} + \ldots\right)\,[0] \qquad (90)$$

In the particular case that $n = m$

$$f_2(z,m,m) = m^2\left(\frac{(m-1)!^2}{0!\,0!}\, j^{-2m} - \frac{2(m-1)!\,(2m-1)!}{0!\,1!}\, j^{-3m} + \ldots \right) [0] \tag{91}$$

In the same way the density functions of the sums of 4,8,16 variates are easily obtained.

For the particular case $m = 2$ we thus have

$$\begin{aligned} f(z,2,2) &= 2(j^{-2} - 2.3\, j^{-4} + 3.4.5\, j^{-6} - 4.5.6.7\, j^{-8} + \ldots) [0] \\ f_2(z,2,2) &= 4(j^{-4} - 2^2 . 3\, j^{-6} + (2.3.4.5 + 2^2 . 3^2) j^{-8} - \\ f_4(z,2,2) &= 16(j^{-8} - 2^3 . 3\, j^{-10} + (2^2 . 3.4.5 + 2^4 . 3^2)\, j^{-12} - \\ f_8(z,2,2) &= 256(j^{-6} - 2^4 . 3\, j^{-18} + (2^3 . 3.4.5 + 2^8 . 3^2)\, j^{-20} - \end{aligned} \tag{92}$$

and transformed into power series

$$\begin{aligned} f(z,2,2) &= (2z - 2z^3 + z^5 - z^7/3 + \ldots)\, z^{+0} \\ f_2(z,2,2) &= 2z^3/3 - 2z^5/5 + 13z^7/105 - \\ f_4(z,2,2) &= 16z^7/7! - 276z^9/9! + \\ f_8(z,2,2) &= 256z^{15}/15! - \\ f_{16}(z,2,2) &= 256^2 z^{31}/31! - \end{aligned} \tag{93}$$

The power series of the cumulative distribution functions are

$$\begin{aligned} F(z,2,2) &= z^2 - z^4/2 + z^6/6 - z^8/24 + \\ F_2(z,2,2) &= 0.16667\, z^4 - 0.06667\, z^6 + 0.01548\, z^8 - \\ F_4(z,2,2) &= (z^8/2520)(1 - 0.26667\, z^2 + 0.03838\, z^4 - \end{aligned} \tag{94}$$

The approximative distributions of the sums can with advantage be computed by means of the formula (76) or (77). Using the first term only, which covers with good approximation the distribution up to $x = 0.5$, we have from

$$f(z,2,2) = 1.5748\, j^{-1.9143}$$

$$f_2(z,2,2) = 1.5748^2\, j^{-3.8286} = 2.4800\, j^{-3.8286} \tag{95}$$

$$F_2(z,2,2) = 2.4800\, j^{-4.8286} = 0.13333\, z^{3.8286} \tag{96}$$

Comparing the values of this equation, called Appr.2, with those computed from the three-term series above, it is found from the table that the difference is negligible, even if only one term has been used. The difference are at most 0.003 %.

z	3 terms	Appr.2
.0	.00 000	.00 000
.1	.00 002	.00 002
.2	.00 026	.00 028
.3	.00 130	.00 133
.4	.00 400	.00 399
.5	.00 936	.00 938

7.3 Addition of a Gamma and a Weibull variate

As an illustration the two variates $Z(2,1)$ and $Z(2,2)$ with the density functions below will be chosen

$$f(z,2,1) = (j^{-2} - 2j^{-3} + 3j^{-4} - 4j^{-5} + \dots)\ [0] \qquad (97)$$

$$f(z,2,2) = (2j^{-2} - 12j^{-4} + 120\ j^{-6} - 1680\,j^{-8} + \dots)\ [0] \qquad (98)$$

Hence

$$f_2(z) = (2j^{-4} - 4j^{-5} - 6j^{-6} + 16j^{-7} + 44j^{-8} - 204j^{-9} + \dots)\ [0]$$

$$= (z^3/3 - z^4/6 - z^5/20 + z^6/45 + z^7/53.6 - z^8/197.6 + \dots)z^{+0} \qquad (99)$$

8. Decomposition of sums of independent variates

The following problem may arise: The distribution of a sum or a product of two independent variates is known and that of one of the components. It is required to find the distribution of the other component.

This problem can be solved by means of "inverse components", denoted by X_+ and X_x and defined by

$$X + X_+ = 1[0] \quad \text{and} \quad X \cdot X_+ = 1[1] \qquad (100)$$

It may be observed that the inverse components of

$$X = k\,j^{-m}[b] \text{ are } X_+ = j^m[-b]/k \quad \text{and} \quad X_x = j^m[1/b]/k \qquad (101)$$

Some inverse addenda will now be determined

8.1 Inverse addenda of Gamma variates

If the mass $1[0]$ is divided by the series

$$f(z,r,1) = j^{-r} - r \cdot j^{-(r+1)} + r(r-1)\, j^{-(r+2)}/2 - \ldots)\ [0]$$

we will obtain the density function of $Z_+(r)$, being

$$f_+(z,r,1) = j^{r} + r\, j^{r-1} + r(r-1)\, j^{r-2}/2! + r(r-1)(r-2) j^{r-3}/3!\ )\ [0] \qquad (102)$$

It is readily seen that this series is finite for any integer r. For example, if $r = 2$, all terms except the first three will disappear.

We have

$$\begin{aligned} r = 1 \quad & f_+(z,1,1) = (j+1)\ [0] \\ r = 2 \quad & f_+(z,2,1) = (j^2 + 2j + 1)\ [0] \\ r = 3 \quad & f_+(z,3,1) = (j^3 + 3j^2 + 3j + 1)\ [0] \end{aligned} \qquad (103)$$

and generally

$$f_+(z,r,1) = (j+1)^r\ [0] \qquad (104)$$

This formula proves that each time a component with the density function $(j+1)\ [0]$ is added to $Z(r,1)$, the parameter of the resulting variate will be reduced by one unit until after r repeated additions the variate becomes $1[0]$.

8.2 Inverse addenda of Weibull variates

Dividing the mass $1[0]$ by the integral series of $f(z,m,m)$, the inverse addendum $Z_+(m,m)$ is obtained. For example, the density function

$$f(z,2,2) = (2j^2 - 12j^{-4} + 120j^{-6} - \ldots)\ [0]$$

corresponds to

$$f_+(z,2,2) = (j^2/2 + 3 - 12\,j^{-2} + 168\,j^{-4} - \dots)\ [0] \tag{105}$$

It is of interest to note that this density function is composed not only of positive and negative multiplex masses but also of a real mass $3[0]$.

8.3 Decomposition of a sum of a Gamma and a Weibull variate

As an illustration take the sum $Z(2,1) + Z(2,2)$, the power and integral series of which are given by (99).

If the inverse variate $Z_+(2,1)$ is added to this sum, we have

$$(2\,j^{-4} - 4\,j^{-5} - 6\,j^{-6} + 16\,j^{-7} - \dots)(j^2 + 2\,j + 1) = (2j^{-2} + 0.j^{-3} - 12j^{-4} + 0\,.\,j^{-5} + 120\,j^{-6} = f(z,2,2) \tag{106}$$

In the same way, if the inverse addendum $Z_+(2,2)$ is added to the sum, the variate $Z(2,1)$ will result.

Table I. The function x^a for small values of a

x	$x^{0.1}$	$x^{0.001}$	x^{+0}	x^0	$x^{-0.001}$	$x^{-0.1}$
-0.001	0.000	0.000	0.000	1.000	0.000	0.000
0.000	0.000	0.000	0.000	1.000	∞	∞
0.001	0.501	0.993	1.000	1.000	1.007	1.995
0.010	0.631	0.995	1.000	1.000	1.005	1.585
0.100	0.794	0.998	1.000	1.000	1.002	1.259
1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.000	1.072	1.001	1.000	1.000	0.999-	0.935

Table II. Exact and approximate values of the distribution function $F(z,2,2)$

z	Exact values	Series (75)			Series (77)	
		1 term	3 terms	5 terms	1 term	3 terms
.0	.0000	.0000	.0000	.0000	.0000	.0000
.1	.0100	.0100	.0100	.0100	.0104	.0104
.2	.0392	.0400	.0392	.0392	.0392	.0392
.3	.0860	.0900	.0860	.0860	.0852	.0852
.4	.1478	.1600	.1478	.1478	.1478	.1478
.5	.2212	.2500	.2212	.2212	.2265	.2224
.6	.3024	.3600	.3029	.3024	.3212	.3024
.7	.3874	.4900	.3896	.3874	.4314	.3862
.8	.4728	.6400	.4789	.4729	.5571	.4728
.9	.5552	.8100	.5705	.5555	.6979	.5561
1.0	.6322	1.0000	.6667	.6333	.8539	.6322
1.1	.7018	1.2100	.7732	.7055	1.0248	.7009
1.2	.7630	1.4400	.9009	.7733	1.2105	.7619
1.3	.8154	1.6900	1.0664	.8414	1.4110	.8154
1.4	.8592	1.9600	1.2941	.9203	1.6261	.8609
1.5	.8940	2.2500	1.6172	1.0299	1.8556	.8981

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13. ABSTRACT

The algebra published in Technical Report No. ASD-TDR 63-63 has been further developed, and its use has been illustrated by some worked examples. After some modifications of the notations, the differentiation and integration of stochastics, including the variates as a special case, have been more thoroughly examined, in particular with respect to the concept of broken derivatives and integrals. A generalized distribution function has been set up. By proper specification of its two shape parameters, it can be brought to reproduce the density functions of the Exponential, Gamma, Pearson Type III, Chi-square, Rayleigh, Weibull, and some more distributions of practical importance. This general function has been expanded in a power series which is transformed in a series, called the integral series. Based on these formulae, rules for summation and multiplication of independent variates are presented and applied to some distributions. Inverse addenda for various variates have been developed and used for decomposition of sums of Gamma and Weibull variates.

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