2. Notations for discontinuous functions

Any variate bounded from below or from above has a density function which, if not itself, at least one of its derivatives has a discontinuity at the bound. If the derivative of order n is discontinuous, but the function and all derivatives of lower orders are continuous, then the function will be said to have a discontinuity of the order n. For example, the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ k.x^2 & \text{if } x \geq 0 \end{cases}$$

and its first derivative are continuous, while its second derivative has a positive jump equal to $2k$ at $x = 0$. Thus the function has a discontinuity of the second order.

Some notations of more general applicability than those previously proposed (l.c.) will now be presented.

For any real numbers x and b the symbol $(x-b)^{+0}$ will be defined by

$$(x-b)^{+0} = \lim_{a \to 0}(x-b)^a = \begin{cases} 0 & \text{if } x \leq b \\ 1 & \text{if } x > b \end{cases} \tag{1}$$

Care must be taken to distinguish this symbol from $(x-b)^0$ which is defined by

$$(x-b)^0 = 1 \tag{2}$$

for any value of x.

We may also use the notation

$$(x-b)^{m+0} = (x-b)^m . (x-b)^{+0} \tag{3}$$

where m is any real number, by which

$$x . (x-b)^{+0} = (x-b+b) . (x-b)^{+0} = (x-b)^{1+0} + b . (x-b)^{+0} \tag{4}$$

In a similar way, the function $x^n . (x-b)^{+0}$ can be written as a sum of $(n+1)$ terms, discontinuous at $x = b$.

Further, for $p \geq 0$,

$$((x-b)^{m+0})^p = (x-b)^{p.m+0} = (x-b)^{p.m} . (x-b)^{+0} \tag{5}$$

It should be noted that $(x-b)^m \cdot (x-b)^0$ must not be written $(x-b)^{m+0}$, since

$$(x-b)^m \cdot (x-b)^0 = (x-b)^m \neq (x-b)^{m+0} \tag{6}$$

The function

$$f(x)\left((x-b)^{+0}-(x-c)^{+0}\right) = \begin{cases} 0 & \text{if} \quad x \leq b \\ f(x) & \text{if} \quad b > x \leq c \\ 0 & \quad\quad c > x \end{cases} \tag{7}$$

is bounded both from below and from above and has a discontinuity of some order at the two bounds. For example, for $f(x) = k \cdot x$ the function (7) is discontinuous with a negative jump equal to $k \cdot c$ at c, and it has a discontinuity of the first order with a positive jump equal to k at b.

In order to avoid confusion, the power series of e^{-x} will be defined by

$$e^{-x} = x^0/0! - x/1! + x^2/2! - \cdots \tag{8}$$

which implies, as a consequence of (5), that

$$e^{-x^{1+0}} = e^{-x} \cdot x^{+0} \tag{9}$$

Applying these notations to the density function of the standardized Gamma variate, defined by

$$f(z) = \begin{cases} 0 & \text{if} \quad z \leq 0 \\ z^{r-1} \cdot e^{-z} & \text{if} \quad z > 0 \end{cases} \tag{10}$$

it may be written alternatively

$$f(z) = z^{r-1+0} \cdot e^{-z} = z^{r-1} \cdot e^{-z} \cdot z^{+0} \tag{11}$$

In the same way, the density function of the standardized Weibull variate may be alternatively written

$$f(z) = m \cdot z^{m-1+0} \cdot e^{-z^m} = m \cdot z^{m-1} \cdot e^{-z^m} \cdot z^{+0} \tag{12}$$

3. Notations for variates and other stochastics

Let the symbol

$$X = \Sigma k_i [x_i] \qquad i = 1, 2, 3, \ldots \tag{13}$$

represent an at most denumerable set of ordered pairs of real numbers, of which x_i may denote a point on the real x-axis in an n-dimensional space R_n and k_i a number associated to this point. It is convenient to visualize k_i as a mass distribution on the x-axis.

If $x_i + \Delta z$ is a point in R_n at a distance Δz in the z-direction from the point x_i, then

$$X = \Sigma k_i [x_i + \Delta z] \tag{14}$$

represents a discrete mass distribution that has been as a whole shifted the distance Δz from its initial position (13).

Instead of having the mass content concentrated in discrete points, the mass may be continuously distributed over parts of the x-axis, which will be denoted by

$$X = p(x)[x] \tag{15}$$

This symbol implies that an infinitesimal mass $p(x)dx$ is associated to each infinitesimal interval dx. It may seem extravagant to repeat the letter x and to enclose it within two different pairs of brackets, but the members within the brackets must not necessarily be the same and this notation offers some advantages, as will be found in the sequel. It emphasizes the idea that the function $p(x)$ is a single object that may be shifted in the space R_n from one position to another with or without deformation.

For example, the symbol

$$X = p(x)[x + \Delta z] \tag{16}$$

represents a displacement of the mass distribution (15) a distance Δz, while the symbol

$$X = p(x)[k \cdot x] \tag{17}$$

represents a deformation of the same distribution.

It is clear that

$$X = p(x)[x] = p(g(x))[g(x)] \tag{18}$$

provided that the function $g(x)$ is real-valued, finite and uniquely defined for all real x.

Thus

$$X = p(e^x)[x] = p(x)[\log x] \tag{19}$$

The function $g(x)$ may also be used to define a correspondence between two objects X and Y, denoted by

$$Y = g(X) \tag{20}$$

Considering that the cumulative distribution function of Y

$$\mathrm{Cdf}(Y) = F_y(y) = F_y(g(x)) = F(x) = \mathrm{Cdf}(X) \tag{21}$$

it follows that the density function of Y

$$\mathrm{Df}(Y) = f(y) = f(x)/g'(x) \tag{22}$$

and consequently

$$Y = (f(x)/g'(x))[g(x)] \tag{23}$$

As an illustration take the important concept of the logarithm of a variate

$$Y = \log X = x \,.\, f(x)[\log x] = e^y \,.\, f(e^y)[y] \tag{24}$$

In the general case, the density function of X is composed of discrete masses and continuously distributed masses. The discrete masses k_i may be real and/or "multiplex" masses, a concept that will be demonstrated and defined in the following chapter.

On this condition

$$X = f(x)[x] = \Sigma k_i[x_i] + p(x)[x] \tag{25}$$

will be called a stochastic, and $f(x)$ its density function.

The variate is an important special case defined by the conditions that k_i is a positive real number, $p(x)$ a real-valued, non-negative, function and

$$\Sigma k_i + \int_{-\infty}^{\infty} p(x)dx = 1 \qquad (26)$$

Then the mass can be interpreted as a probability and the symbol (25) represents a variate (random variable), which implies that the x-values within the square brackets represent anyone of the values the variate can take and

$$k_i + p(x)dx$$

the probability of choosing at random a value of X belonging to the interval dx, where k_i is a finite probability, if any, belonging to dx.

Thus, the symbol

$$X = 1[0] \qquad (27)$$

implies that there is 100% probability of X taking the value 0.

4. Differentiation and integration of stochastics

A mathematical operator, denoted by j_z and applied to a stochastic X, will be defined by

$$X \cdot j_z = f(x) \cdot j_z[x] \qquad (28)$$

where

$$j_z[x] = \lim_{\Delta z \to 0}(1[x] - 1[x + \Delta z]) / \Delta z$$

This limiting process will, for brevity, be denoted by

$$j_z[x] = (1[x] - 1[x + dz])/dz \qquad (29)$$

where $dz = \Delta z \to 0$ is an infinitesimal number.

The object $X . j_z$ will be called the derivative of X. In particular, if $dz = dx$, the subscript will be dropped, when no confusion can arise, and

$$f(x) . j[x] = f(x)[x]/dx - f(x)[x+dx]/dx \tag{30}$$

The operator j_z may be applied n succesive times, which will be denoted by $f(x) . j_z^n$.

Putting

$$m = n + a \geq 0 \quad (n = \text{pos.integer}; \ 0 \leq a \leq 1) \tag{31}$$

also the operator j_z^m will be introduced and its inverse j_z^{-m}, defined by

$$(X j_x^m) j_z^{-m} = X \tag{32}$$

The limiting process (30) will now be applied to the three alternatives, viz., discrete, continuous, and discontinuous mass distributions.

4.1 Discrete mass distributions

By (29) it follows that the derivative of $X = \Sigma k_i [x_i]$ is

$$X . j = \Sigma k_i . j[x_i] \tag{33}$$

that is, the operator j has to be applied to each term. Further, each term $k_i . j[x]$ can be interpreted as a "duplex mass" composed of one infinitely large mass $1/dx$ located at $x = x_i$ and one infinitely large, negative mass $-1/dx$ located at the infinitesimal distance dx from the positive mass.

Another interpretation of the object $k_i j[x_i]$ is to think of k_i, not as a mass, but as a force equal to k_i kgf. Then $k_i . j$ may be interpreted as a finite moment equal to k_i kgf.cm.

Repeating this process, the duplex mass is transformed into a "triplex mass" $k_i . j^2$, composed of three infinitely large masses $1/dx^2$, $-2/dx^2$, $1/dx^2$, located at the infinitesimal distance dx from each other.

In the same way, the symbol $k\,j^n$ may be interpreted as a "multiplex mass" with $(n+1)$ components, the sum of which is equal to zero.

If now the inverse operator j^{-r}, where r is a positive integer, is applied to j^n, we have by definition

$$k \cdot j^n \cdot j^{-r}[x_i] = k \cdot j^{n-r}[x_i] \tag{34}$$

provided that r n. By this operator the multiplex mass with $(n+1)$ components is reduced to a multiplex mass with $(n+1-r)$ components.

In a following paragraph this formula will be generalized to cover the cases that n and r are substituted by arbitrary numbers.

It may be added that, in analogy with the preceding, the symbol

$$X \cdot j_z = f(x) \cdot j_z[x] \tag{35}$$

may be interpreted as two separate continuous mass distributions with infinitely large density functions $f(x)/dz$ and $-f(x)/dz$, located at the infinitesimal distance dz from each other. For $dx = dz$ a single mass distribution over the x-axis will result, as will be demonstrated below.

4.2 Continuous mass distributions

Let

$$X = f(x)[x] \tag{36}$$

be a stochastic, defined by its density function $f(x)$ and

$$X j = f(x) j[x] \tag{37}$$

its derivative. From the limiting process (30) it follows, on the condition that $f(x)$ is everywhere continuous, that

$$X j = f'(x)[x] \tag{38}$$

where

$$f'(x) = d\,f(x)/dx$$

is identical with the ordinary derivative of a continuous function.

From this identity between j and d/dx it can be concluded that for continuous functions

$$(f(x) \,.\, g(x))j = f(x) \,.\, (g(x)j) + (f(x)j)g(x) \qquad (39)$$

This rule does not apply to the operator j^n.

Repeating this process we have

$$X \,.\, j^n = f^{(n)}(x)[x] \qquad (n = \text{pos. integer}) \qquad (40)$$

It is obvious that, if X is a variate, then Xj is not, since its total mass is equal to zero.

Let us now apply the operator j^{-1} to each side of (38). Then

$$(Xj)j^{-1} = X = f'(x)\,j^{-1}[x] = f(x)[x] \qquad (41)$$

Hence the operator j^{-1} applied to a continuous function is identical with the ordinary integration process. In the same way j^{-n} implies this process repeated n times.

The operator j^n will now be generalized to the case that n is substituted by a real number $m \geq 0$, or

$$f(x) \,.\, j^m = d^m f(x)/dx^m \qquad (42)$$

If it is required that for any positive real numbers $(a,b) \geq 0$

$$(f(x)j^a)j^b = f(x) \,.\, j^{a+b} = (f(x)j^b)j^a \qquad (43)$$

then it is readily seen that, if $f(x) = k \,.\, x^m$, the formula

$$x^m \,.\, j^b = d^b(x^m)/dx^b = m!\, x^{m-b}/(m-b)! \quad (b \quad m+n) \qquad (44)$$

satisfies the required condition (43) and, further, that this formula reduces to the ordinary rules of differentiation, if b is a positive integer.

The restricting condition b m + n is motivated by the following reasoning:

By (44) we have

$$x^m . j^{m+a} = m!\, x^{-a}/(-a)! \quad (0 \leq a \quad 1) \tag{45}$$

The same result is obtained, also if the differentiation is performed in two steps, as

$$x^m j^{m+a} = m!\, x^0 . j^a = m!\, x^{-a}/(-a)! \tag{46}$$

It is of interest to note that the derivative j^a of the constant mass distribution x^0 does not disappear.

For $a = 1$ we have

$$x^m . j^{m+1} = m!\, x^0 . j = 0 \tag{47}$$

since, by substituting x^0 for $f(x)$ in (30), it follows that $x^0 . j = 0$.

But according to (44), we have for $a = 1$

$$x^m . j^{m+1} = m!\, x^{-1}/(-1)! \tag{48}$$

Considering that $(-1)! = \infty$, the right-hand member is equal to zero for any finite value of x, but for $x = 0$ it takes the indeterminate form ∞/∞.

In order to determine this value, the mass density function

$$y = x^{-a}/(-a)! \tag{49}$$

will be examined. The mass content of the interval $(0,x)$

$$M = \int_0^x x^{-a}\, dx/(-a)! = x^{1-a}/(1-a)! \tag{50}$$

thus becomes $M = 1$ for $a = 1$, independently of x, and it can be concluded that

$$x^{-1}/(-1)! = 1[0] \tag{51}$$

This result differs from equ.(47) and motivates the restriction on equ.(44), which may be completed by the formula

$$x^m \cdot j^{m+n} = 0 \qquad (n = \text{pos.integer}) \tag{52}$$

obtained by multiplying each member of (47) by j^{n-1}.

In the same way, multiplying equ.(44) by j^{-b} and substituting $(m+b)$ for m we have

$$x^m \cdot j^{-b} = m! \cdot x^{m+b}/(m+b)! \tag{53}$$

which proves that equ.(44) is valid also for negative values of b.

4.3 Discontinuous mass distributions

Applying the limiting process (30) to the discontinuous function $(x-b)^{+0}$, it is easily found that

$$k(x-b)^{+0} \cdot j = k[b] \tag{54}$$

and thus

$$((x-b)^{+0} - (x-c)^{+0})\, j = 1[b] - 1[c] \tag{55}$$

Repeating the process (54) it follows that

$$k(x-b)^{+0}\, j^m = k \cdot j^{m-1}[b] \tag{56}$$

which, for $(m-1)$ = positive integer, represents a multiplex mass located at $x = b$. The interpretation of (56) for arbitrary values of m will be demonstrated in the following.

In the same way, by (30) we have

$$f(x)((x-b)^{+0} - (x-c)^{+0}) \cdot j[x] = f(b)[b] + f'(x)((x-b)^{+0} - (x-c)^{+0})[x] - f(c)[c] \tag{57}$$

and repeating this process, omitting for brevity the second term,

$$f(x)(x-b)^{+0} \cdot j^n[x] = (f(b)j^{n-1} + f'(b)j^{n-2} + \dots + f^{(n-1)}(b))[b] + f^{(n)}(x)((x-b)^{+0} \tag{58}$$

In the particular case that $f^{(n)}(x)$ and higher derivatives are equal to zero, or can be neglected, the derivative of a discontinuous function can be expanded in a series of discrete multiplex masses located at the bounds. As an illustration take the function $f(x) = x^2$. Then by equ.(58)

$$x^2((x-b)^{+0} - (x-c)^{+0})\, j^3 = (b^2 j^2 + 2b\, j + 2)[b] - (c^2 j^2 + 2c\, j + 2)[c] \tag{59}$$

The function itself can be expanded in a series of multiple integrals j^{-n}. For example, multiplying equ.(59) by j^{-3} it follows that

$$x^2((x-b)^{+0} - (x-c)^{+0}) = (b^2 j^{-1} + 2b j^{-2} + 2b j^{-2} + 2 j^{-3})\,[b] - (c^2 j^{-1} + 2c j^{-2} + 2 j^{-3})[c] \tag{60}$$

The series (58) takes a particularly simple form, when $f(x)$ and its derivatives up to a certain order are equal to zero at the bounds, as will be illustrated by means of the function $(x-b)^{m+0}$. By equ.(58) then, provided $m \geq n$,

$$(x-b)^{m+0} \,.\, j^n = m!\,(x-b)^{m-n+0}/(m-n)! \tag{61}$$

This rule applies also to the broken derivative j^a and consequently also to j^m as demonstrated by comparing the direct derivative and that one performed in two steps:

$$(x-b)^{m+0} \,.\, j = m(x-b)^{m-1+0} \quad (m \geq 0)$$

but also

$$((x-b)^{m+0} j^a)\, j^{1-a} = m!\,(x-b)^{m-a+0} \,.\, j^{1-a}/(m-a)! = m(x-b)^{m-1+0}$$

Hence

$$(x-b)^{m+0} \,.\, j^{m+1} = m!\,(x-b)^{+0} \,.\, j = m![b] \tag{62}$$

and, after multiplication by $j^{-(m+1)}$

$$(x-b)^{m+0} = m!\, j^{-(m+1)}[b]$$

and

$$j^{-m}[b] = (x-b)^{m-1+0}/(m-1)! \qquad (m \geq 0) \tag{63}$$

By use of these two formulae any analytic function bounded from below can, after expansion in a power series (sometimes finite), be transformed into a series of multiple integrals j^{-m}, and, reversely, any such series of integrals can be transformed into a power series. The use of the integral series depends on the fact that each term $j^{-m}[b]$ can be considered a mass j^{-m} located at $x=b$ and treated as such, when used for composition of variates and for determination of inverse variates, as will be demonstrated in the sequel.

The two symbols $j^{-a}[0]$ and $j^{a}[0]$ for $0 \leq a \leq 1$ and their geometrical representation will now be more closely examined.

By (63) we have

$$j^{-a}[0] = x^{a-1+0}/(a-1)! \tag{64}$$

This function tends to infinity as x tends to zero, but in such a way that the mass content of the interval $(0,x)$, being

$$M = x^{a}/a! \tag{65}$$

tends to zero with x, that is, there is no finite mass concentrated in $x=0$. Thus, if $a \to 0$ then $j^{-a}[0] \to 1[0]$ in accordance with equ.(51). On the other hand, if $a \to 1$, then $j^{-a}[0] \to x^{+0}$. Some values of x^{a} for small values of $\underline{a}$ are presented in Table I.

The symbol $j^{a}[0]$ has, if $a=n=$pos.integer, already been identified with a multiplex mass located in $x=0$. It will, in general, be defined as the derivative of $j^{a-1}[0]$ or

$$j^{a}[0] = (j^{a-1}[0])j = x^{-a+0} . j/(-a)! \tag{66}$$

Applying the limiting process (30) to the function $x^{-a}/(-a)!$ it will be found that $j^{a}[0]$ is composed of one infinitely large, positive mass $1/(1-a)!\,dx^{a}$ concentrated in $x=0$ and one equally large but negative mass, distributed to the right of $x=dx$ and with an infinite density at this point. When $a \to 1$ the negative mass becomes more and more concentrated in $x=dx$, that is, j^{a} tends to a duplex mass as $a \to 1$.