COMPOSITION AND DECOMPOSITION OF BOUNDED VARIATES WITH SPECIAL REFERENCE TO THE GAMMA AND THE WEIBULL DISTRIBUTIONS

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FOREWORD

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ABSTRACT

The algebra published in Technical Report No. ASD-TDR 63-63 has been further developed, and its use has been illustrated by some worked examples. After some modifications of the notations, the differentiation and integration of stochastics, including the variates as a special case, have been more thoroughly examined, in particular with respect to the concept of broken derivatives and integrals. A generalized distribution function has been set up. By proper specification of its two shape parameters, it can be brought to reproduce the density functions of the Exponential, Gamma, Pearson Type III, Chi-square, Rayleigh, Weibull, and some more distributions of practical importance. This general function has been expanded in a power series which is transformed in a series, called the integral series. Based on these formulae, rules for summation and multiplication of independent variates are presented and applied to some distributions. Inverse addenda for various variates have been developed and used for decomposition of sums of Gamma and Weibull variates.

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1. Introduction

The algebra presented in an earlier publication:"Outline of an algebra of stochastic quantities", Tech.Docum.Rep.No.ASD-TDR-63-63, has been further developed and its use is, in this report, illustrated by applications to the Gamma and the Weibull distributions.

Earlier notations for discontinuous functions have been modified in order to make them suitable also for distributions bounded from above, as prsented in the second chapter.

Notations for stochastics, including variates, have earlier been proposed. Their usefulness is indicated in the third chapter.

Differentiation and integration of stochastics are defined in the fourth chapter. The properties of the symbols j^m and j^{-m} are more thouroughly examined than in earlier publications and some new formulae have been derived and are demonstrated.

A generalized distribution function, involving two shape parameters, is presented in the fifth chapter. By proper specification of these parameters, it reproduces the density functions of the following distributions: Exponential, Gamma, Pearson Type III, Chi-square, Sampling distribution of standard deviations, Rayleigh, Weibull, and some more distributions.

By use of the formulae presented in the fourth chapter, the generalized density function has been expanded in a power series and transformed into an integral series, as demonstrated in the sixth chapter. This general formula is then specified for the Exponential, Gamma, and Weibull variates.

Rules for composition of independent variates are presented in the seventh chapter as well as their applications to the abovementioned variates.

Finally, decomposition of sums of independent variates by means of inverse variates is demonstrated in the eighth chapter. Inverse addenda for the Exponential, Gamma, and Weibull variates are given. They have proved to be particularly simple for the Exponential and the Gamma distributions. Applications to a sum of a Gamma and a Weibull variate is presented for illustration purposes.