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FOREWORD

This report was prepared by Prof. Dr. Waloddi Weibull, Lausanne, Switzerland, under USAF Contract No. AF 61(052)-522. The contract was initiated under Project No. 7351, "Metallic Materials," Task No. 735106, "Behavior of Metals." The contract was administered by the European Office, Office of Aerospace Research. The work was monitored by the Air Force Materials Laboratory, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, under the direction of Mr. W. J. Trapp.

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This technical report has been reviewed and is approved.

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ABSTRACT

The classical method of moments for estimating distribution parameters, which consists in equating as many of the population moments as the number of unknown parameters to the corresponding sample moments, has been much appreciated, because it is quite easy to use and does not need any ordering of the observations. However, in some cases its efficiency is very poor, so it has to be used with some precaution. In order to elucidate this statement, formulas for the asymptotic efficiency of the most used estimators have been derived for the alternatives of one, two and three unknown parameters. Numerical values corresponding to several values of α and for the cases of one or two unknown parameters have been computed and are presented.

INTRODUCTION

The method of moments is the classical method for estimating distribution parameters from samples randomly drawn from a population. Its principle ooneiats in equating a8 many of the population moments as the number of unknown parameters to the corresponding sample moments. The numerical computations of the parameters are quite simple and no ordering of the observations is needed. However, in some oases the efficiency of the estimators is very poor, so the method has to be used with some precaution. Furthermore, it cannot be applied to truncated and censored samples.

The Population Moments

Writing-the distribution function in the form

$$P - 1 - e^{-((x-\mu)/\beta)^{1/\alpha}} \tag{1}$$

where μ - location parameter, β - scale parameter, and a-shape parameter, we have two types of moments:

moments about origin	and	oentral moments
$\alpha_1 = \mu + \beta \cdot g_1$		$\mu_1 = 0$
$\alpha_2 = \mu^2 + \beta^2 \cdot g_2 + 2\mu\beta \cdot g_1$		$\mu_2 = \beta^2(g_2 - g_1^2)$
. . . . . .		$\mu_3 = \beta^3(g_3 - 3g_1g_2 + 2g_1^2)$
where $g_v = (v\alpha)!$		$\mu_4 = \beta^4(g_4 - 4g_1g_3 + 6g_1^2g_2 - 3g_1^4)$

(2)

Thus g_v is a function of the shape parameter a only, and α_1 la the population mean and μ_2 la the population variance.

The Sample Moments

These moments are functions of the observations only and explicitly independent Qf the distribution in question. We have two types of moments:

moments about origin	and	central moments	
$a_1 = \Sigma x_i/n = \bar{x}$		$m_1 = 0$	
$a_2 = \Sigma x_i^2/n$		$m_2 = \Sigma(x_i - \bar{x})^2/n$	
. . . .		$m_3 = \Sigma(x_i - \bar{x})^3/n$	(3)
$a_v = \Sigma x_i^v/n$		$m_v = \Sigma(x_i - \bar{x})^v/n$	

Here a_1 is the sample mean and m_2 is the sample variance.

For any sample moment we have

$$E(a_v) = \alpha_v \quad \text{and} \quad Var(a_v) = (\alpha_{2v} - \alpha_v^2)/n \qquad (4)$$

In particular

$$E(\bar{x}) = \alpha_1 = \mu + \beta \cdot g_1$$
$$Var(\bar{x}) = \mu_2/n = \beta^2(g_2 + g_1^2)/n \qquad (5)$$

In general

$$E(a_v - \alpha_v)^{2k-1} = O(1/n^k) \; ; \; E(a_v - \alpha_v)^{2k} = O(1/n^k) \qquad (6)$$

For the central sample moments it follows that

$$E(m_1) = 0 \qquad Var(m_1) = 0$$
$$E(m_2) = (1 - 1/n)\mu_2 \qquad Var(m_2) = (\mu_4 - \mu_2^2)/n - 2(\mu_4 - 2\mu_2^2)/n^2$$
$$E(m_3) = (1 - 3/n + 2/n^2)\mu_3 \qquad + (\mu_4 - 3\mu_2^2)/n^3$$

In general (7)

$$E(m_v) = \mu_v + O(1/n) \qquad Var(m_v) = (\mu_{2v} - 2v\mu_{v-1} \cdot \mu_{v+1} - \mu_v^2 + v^2\mu_2\mu_{v-1}^2)/n + O(1/n^2)$$

As mentioned above, the number of different moments required for estimation of the parameters is equal to the number of unknown parameters. We will now examine the various alternatives.

One unknown parameter

In this case only one population moment has to be equated to the corresponding sample moment, that is, moment estimators can be obtained by putting either $a_v = \alpha_v$ or $m_v = \mu_v$, where v is arbitrary.

There are three alternatives to examine.

Location parameter μ unknown (α and β known)

Looking at equs.(2) we observe that μ is not involved in the expressions of the central moments, and further that the moments of higher order than the first one will lead to rather complicated formulae. Therefore, the only estimator of practical interest will be that one obtained by equating the population and the sample means, that is,

$$\mu + \beta \cdot g_1 = \bar{x}$$

or, denoting the moment estimator of μ by μ^*,

$$\mu^* = \bar{x} - \beta \cdot g_1 \tag{8}$$

Since α and β are assumed to be known, the quantity $\beta \cdot g_1 = \beta \cdot \alpha!$ is a known constant.

From equs.(5) it follows that

$$E(\mu^*) = \mu + \beta g_1 - \beta g_1 = \mu \tag{9}$$

that is, μ^* is an unbiased estimate of μ.

Further,

$$\mathrm{Var}(\mu^*) = \mathrm{Var}(\bar{x}) = \beta^2(g_2 - g_1^2)/n \tag{10}$$

From TR-4, Appendix A, p.3, the lowest variance of any estimate of μ (a and β known) attainable is:

$$\begin{aligned} \mathrm{Var}(\hat{\mu}) &= \alpha^2\beta^2(1 - 2\alpha)/(1 - \alpha)^2(1 - 2\alpha)!\, n & 0 \le \alpha < 0.5 \\ &= \beta^2/n \log n & \alpha = 0.5 \\ &= \beta^2/n^{2\alpha} & 0.5 < \alpha < \infty \end{aligned} \tag{11}$$

(Here $\hat{\mu}$ denotes any efficient estimate)

The asymptotic efficiency of μ^* is defined by

$$\text{Asy Eff }(\mu^*) = \text{Var}(\hat{\mu})/\text{Var}(\mu^*) \tag{12}$$

Hence, for $0 \leq \alpha 40.5$

$$\text{Asy Eff}(\mu^*) = \alpha^2(1 - 2\alpha)/(1 - \alpha)^2(1 - 2\alpha)!(g_2 - g_1^2) \tag{13}$$

while for a-0.5 the asymptotic efficiency is equal to zero, since the efficient estimate in this case is hyper-efficient.

The values of Asy $\text{Eff}(\mu^*)$ have been computed and are presented in Table I.

$$\text{For } a = 0 \qquad \text{Asy Eff}(\mu^*) = \lim_{\alpha \to 0} \alpha^2/ (g_2 - g_1^2) \tag{14}$$

Derivating twice and noting that, according to definition of the Digamma Function

$$\Psi(\alpha) = d \log g_1/d\alpha = g_1'/g_1$$

we obtain

$$g_1' = d g_1/d\alpha = g_1 \cdot \Psi(\alpha) \tag{15}$$

Aftersome calculationit thenfollowsthat

$$\text{Asy Eff}(u^*) = 1/ \Psi'(0) \qquad (a-0) \tag{16}$$

The function $\Psi'(\alpha)$, known as the Trigamma Function, is, together with the Factorial Function and the Digamma Function, computed to twelve decimal places and tabulated in Matematical Tables, Vol I, Brit.Ass. Adv.So., Cambridge, 1931, from which

$$\text{Asy Eff }(\mu^*) = 1/1.644934 = 0.60793 \qquad (\alpha = 0)$$

Scale parameter β unknown (a and μ known)

In this case have two alternatives of interest, viz., either putting al = al or $\mu_2 = m_2$.

The estimator obtained from the first moments will be denoted by β_1^* and from the second central momenta by β_2^*.

From equs (2) and (3) it follows that

$$\beta_1^* = (\bar{x} \ \mu)/g_1 \tag{17}$$

where μ/g_1 is a known constant.

Henceby (5)

$$E(\beta_1^*) = \alpha_1/g_1 - \mu/g_1 = \beta \tag{18}$$

thatis, β_1^* is an unbiased estimate of β with the variance

$$Var(\beta_1^*) = \beta^2(g_2 - g_1{}^2)/g_1^2 \cdot n \tag{19}$$

From TB-4, App.A,p.3, we have

$$Var(\beta) = \alpha^2\beta^2/n \tag{20}$$

and consequently

$$Asy\ Eff(\beta_1^*) = \alpha^2 g_1^2/(g_2 - g_1^2) \tag{21}$$

The values of Asy Eff (β_1^*) have been computed and are presentedin Table I.

From the preceding it is readiseen that

$$Asy\ Eff(\alpha_1^*) = 0.60793 \qquad (\alpha = 0) \tag{22}$$

Fur the second alternative $(m_2 = \mu_2)$ we obtain

$$m_2 = \beta^2(g_2 - g_1^2)$$

and

$$\beta_2^* = [m_2/(g_2 - g_1^2)]^{1/2} \tag{23}$$

It is known (Cf. Cramér [1], p.353) that

$$E(\sqrt{m_2}) = \sqrt{\mu_2} + O(1/n)$$
$$Var(\sqrt{m_2}) = (\mu_4 - \mu_2^2)/4\mu_2 n + O(1/n^2) \tag{24}$$

Hence, for large samples

$$E(\beta_2^*) = [\mu_2/(g_2 - g_1^2)]^{1/2} = \beta \tag{25}$$

that is, β_2^* is an asymptotically unbiased estimate of β.

Further,

$$Var(\beta_2^*) = (\mu_4 - \mu_2^2)/4n\mu_2(g_2 - g_1^2) \tag{26}$$

or

$$Var(\beta_2^*) = \beta^2(g_4 - 4g_1g_3 + 8g_1^2g_2 - 4g_1^4 - g_2^2)/4n(g_2 - g_1^2)^2 \tag{27}$$

and by equ. (20)

$$Asy\ Eff\ (\beta^*{}_2) = 4\alpha^2(g_2 - g_1^2)^2/(g_4 - 4g_1g_3 + 8g_1^2g_2 - 4g_1^4 - g_2^2) \tag{28}$$

Another way of arriving at this result is the following. The estimator

$$(\beta^2)^* = m_2/(g_2 - g_1^2) \tag{29}$$

has, for large samples, a mean

$$E(\beta^2)^* = \mu_2/(g_2 - g_1^2) = {}^2\beta \tag{30}$$

and a variance

$$Var(\beta^2)^* = (\mu_4 - \mu_2^2)/n(g_2 - g_1^2)^2 \tag{31}$$

Considering that for large samples the estimator $(\beta^2)^*$ takes values that almost always are very close to its mean

$$\beta_o^2 = \mu_2/(g_2 - g_1^2) \tag{32}$$

we can uss the approximation

$$\beta^2 = \beta_o^2 + 2\beta_o(\beta - \beta_o) \tag{33}$$

Thus

$$Var(\beta^2)^* = 4\beta_o^2 \,.\, Var(\beta_2^*)$$

or

$$Var(\beta_2^*) = Var(\beta^2)^*/4\beta_o^2 \tag{34}$$

Hence

$$Var(\beta_2^*) = (\mu_4 - \mu_2^2)/4n\mu_2(g_2 - g_1^2) \quad (35)$$

which agrees with equ. (26).

Values of Asy Eff (β_2^*) have been computed and are presented in Table I.

Shape parameter α unknown (β and μ known)

Also in this oaee we will examinetwo alternatives viz., $a_1 = \alpha_1$ and $\mu_2 = m_2$.

In the first alternative, estimator of the quantity gl is easily obtained viz.,

$$g_1^* = (\bar{x} - \mu)/\beta \quad (36)$$

with the characteristics

$$E(g_1^*) = (\alpha_1 - \mu)/\beta = g_1 \quad (37)$$

and

$$Var(g_1^*) = (g_2 - g_1^2)/n \quad (38)$$

For large values of n, the function $g_1 = \alpha!$ of α is almost always very close to the value that corresponds to the true value α_o and thus we can use the approximation

$$g_1 = \alpha_o! + (dg_1/d\alpha_o)(\alpha - \alpha_o) \quad (39)$$

and thus

$$Var(\alpha_1^*) = Var(g_1^*)/(dg_1/da_o)^2 \quad (40)$$

where $dg_1/d\alpha_o$ denotes $(dg_1/d\alpha)_{\alpha = g_o}$

Introduoing (15) into (39) and considering (38)

$$Var(\alpha_1^*) = (g_2 - g_1^2)/n \cdot g_1^2 \cdot \Psi(\alpha) \quad (41)$$

From TR-4, App.A, we have

$$Var(\hat{\alpha}) = 0.548342\,\alpha^2/n \tag{42}$$

and thus

$$Asy\ Eff(\alpha_1^*) = 0.548342\,\alpha^2 g_1^2 . \Psi(\alpha)/(g_2 - g_1^2) \tag{43}$$

From (20) and (43) it follows that

$$Asy\ Eff(\alpha_1^*) = 0.548342\ \Psi(\alpha) . Asy\ Eff(\beta_1^*) \tag{44}$$

Values of Asy $Eff(\alpha_1^*)$ have been computed and are presented in Table I.

Equating the two moments m_2 and μ_2 an estimator of another function of α can be obtained, viz.,

$$G_2^* = (g_2 - g_1^2)^* = m_2/\beta^2 \tag{45}$$

with the characteristics

$$E(G_2^*) = \mu_2/\beta^2 = (2\alpha_o)! - (\alpha_o)!^2 \tag{46}$$

and

$$Var(G_2^*) = (\mu_4 - \mu_2^2)/n . \beta^4 = (g_4 - 4g_1g_3 + 8g_1^2g_2 - 4g_1^4 - g_2^2)/n\beta^4 \tag{47}$$

Since the estimator G_2^* almost always takes values close to the value $(2\alpha_o)! - (\alpha_o)!^2$ we can use the *linear* approximation as in the previous case, and will then arrive at

$$Var(\alpha_2^*) = (g_4 - 4g_1g_3 + 8g_1^2 g_2 - 4g_1^4 - g_2^2)/n(dG_2^*/d\alpha_o)^2 \tag{48}$$

From the definition of the Digamma Function it follows that

$$dg_2/d\alpha = 2g_2 . \Psi(2\alpha) \tag{49}$$

and hence by (15) that

$$dG_2^*\ d\alpha = 2[g_2 . \Psi(2\alpha) - g_1^2 . \Psi(\alpha)] \tag{50}$$

Introducing (50) into (48)

$$Var(\alpha_2^*) = (g_4 - 4g_1g_3 + 8g_1^2g_2 - 4g_1^4 - g_2^2)/4n[g_2\ \Psi(2\alpha) - g_1^2\Psi(\alpha)]^2 \tag{51}$$

and by (42)

$$\text{Asy Eff}(\alpha_2^*) = 2.193368 \cdot \alpha^2[g_2\Psi(2\alpha) - g_1^2\Psi(\alpha)]^2/(g_4 - 4g_1g_3 + 8g_1^2g_2 - 4g_1^4 - g_2^2) \quad (52)$$

Values of Asy Eff(α_2^*) have been oomputed and exe presented in Table I.

Two unknown parameters

In this case it is required to equate two population moments to the corresponding sample moments. For this purpose we will use α_1 and μ_2. There are three alternatives to examine, via., only α known, only μ known, and only β known. Since the third alternative does not appear In any practical oases, it will be omitted.

Shape parameter a known (β and μ unknown)

The estimators are derived from the equations

$$\begin{aligned} \bar{x} &= \mu + \beta \cdot g_1 \\ m_2 &= \beta^2(g_2 - g_1^2) \end{aligned} \quad (53)$$

Since g_2 and g_1 are known, as being functions of the known parameter a, the Value of β can be estimated by use of the second equation in the same way as leading to the estimator β_2^* in equ.(23).

Considering that in this case the variance of the efficient estimator of β differs from equ. (20) and is now, according to TB-4, App.A,p.4, for $\alpha < 0.5$,

$$\text{Var}(\hat{\beta}) = \beta^2(1-\alpha)^2(1 - 2\alpha)!\ \alpha^2/[(1-\alpha)^2(1 - 2\alpha)! - (1-2\alpha)(1-\alpha)!^2]n \quad (54)$$

Hence by (27) and (54)

$$\text{Asy Eff}(\beta^*) = [[(1-\alpha)^2(1-2\alpha)!]/[(1-\alpha)^2(1-2\alpha)!-(1-2\alpha)(1-\alpha)!^2]] \cdot \text{Asy Eff}(\beta_2^*) \quad (55)$$

Values of Asy Eff (β^*) have been oomputed and are presented in Table I.

By eliminating β from the two equs. (53) the estimator for μ becomes

$$\mu^* = \bar{x} - g_1[m_2/(g_2 - g_1^2)]^{1/2} \tag{56}$$

Following the calculations made Dubey [2], we have

$$Var(\mu^*) = \beta^2[4g_2(g_2^2 - g_1 g_3) + g_1^2(g_4 - g_2^2)]/4n(g_2 - g_1^2)^2 \tag{57}$$

The variance of the efficient estimator is, as given in TB-4, App.A, p.4) for $\alpha \leq 0.5$

$$Var(\hat{\mu}) = \alpha^2\beta^2(1 - 2\alpha)/[(1 - \alpha)^2(1 - 2\alpha)! - (1 - 2\alpha)(1 - \alpha)!^2] \tag{58}$$

The Asy Eff(μ^*) is obtained as the ratio of (58) and (57). Values have been computed and are presented in Table I.

Location parameter μ known (α and β unknown)

Without loss of generality, it can be assumed that $\mu = 0$. Then the basic equations become

$$\bar{x} = \beta \cdot g_1$$
$$m_2 = \beta^2(g_2 - g_1^2) \tag{59}$$

+)

The quantity V is known as the coefficient of variation of the sample. Its sampling mean and variance are known (Cf. Cramér [1], p.358) on the condition that the variable takes only positive values. This condition is satisfied, since $x \geq \mu = 0$.

For large samples

$$E(V) = \sqrt{\mu_2}/\alpha_1 = \sqrt{g_2 - g_1^2}/g_1 \tag{61}$$

Thus, the estimator is asymptotically unbiased and Its variance

$$Var(K^*) = [\alpha_1^2(\mu_4 - \mu_2^2) - 4\alpha_1\mu_2\mu_3 + 4\mu_2^3]/4\alpha_1^4\mu_2 \cdot n \tag{62}$$

+) Eliminating β we have

$$K^* = [\sqrt{g_2 - g_1^2}/g_1]^* = \sqrt{m_2}/\bar{x} = V \tag{60}$$

Approximating the funotion K by a linear expression as in the preceding derivations we obtain after some calculations

$$\mathrm{Var}(\alpha^*) = [\alpha_1^2(\mu_4 - \mu_2^2) - 4\alpha_1 \mu_2\mu_3 + 4\mu_2^2]/ 4\alpha_1^4 \mu_2 . n . (dK/d\alpha_o)^2 \tag{63}$$

From TB-4, App.A we have

$$\mathrm{Var}(\hat{\alpha}) = 0.60793 \alpha^2/n \tag{64}$$

and thus

$$\text{Asy Eff}(\alpha^*) = [2.43172\alpha^2\alpha_1^4\mu_2(dK/d\alpha_o)^2/[\alpha_1^2(\mu_4 - \mu_2^2) - 4\alpha_1\mu_2\mu_3 + 4\mu_2 \tag{65}$$

Introducing the formulas (2) into (65) and considering that

$$dK/d\alpha = g_2[\psi(2\alpha) - \psi^2(\alpha)]/g_1 \sqrt{g_2 - g_1{}^2} \tag{66}$$

the values of Asy Eff (a*) can be computed for any value of a.

<u>All three parameters unknown</u>

In this case three moments are required and the most simple ones are α_1, μ_2, and μ_3.

Thus

$$\begin{aligned} \bar{x} &= \mu + \beta \cdot g_1 \\ m_2 &= \beta^2(g_2 - g_1^2) \\ m_3 &= \beta^3(g_3 - 3g_1g_2 + 2g_1^2) \end{aligned} \tag{67}$$

From the last two equations β can be eliminated, giving

$$(g_3 - 3g_1g_2 + 2g_1)/(g_2 - g_1^2)^{3/2} = m_3/m_2{}^{3/2} = G_1 \tag{68}$$

The quantity G_1 is known as the coefficient of skewness of the sample. Its mean and variance are known (Cf. Cramér[1], p.357).

Using the left-hand member, which is function of a only, as an estimator, its asymptotic efficiency can be computed, as indicated in the previous oases.