…ing errors in S.

2. Complete Fatigue Diagram

From the preceding considerations it may be clear that the relations between the three quantities P, S, and N, if known, give a complete description of the fatigue as a phenomenon. It should be observed that such $P - S - N$ relations may be developed without any knowledge about the real nature of fatigue, but they are nevertheless invaluable for the designer, provided they are representative of the material and the mode of stressing in service. From a general point of view they may be regarded as a useful preparation for the arduous task of understanding the basic problems of fatigue failures. As will be demonstrated in the sequel, the relations procure also the means of determining the influence of the dimensions of the specimen.

The easiest way to present the relations is by a $P - S - N$ diagram as shown in Fig. 4, where curves over two of the quantities are drawn for selected constant values of the third. This method evidently offers three possibilities. Field 1 gives the load S as a function of N, i. e. the $S - N$-curves for some values of P between 0 and 1. If we introduce the notation $S(P, N)$ for the load giving a failure after N cycles with the probability P, obviously $U = S(O, N)$ is the lower boundary of all possible values, while $E = S(P, \infty)$ denotes the asymptotic values of the $S - N$ curves. Thus, U is a function of N only and E a function of P only. The quantity U is the most

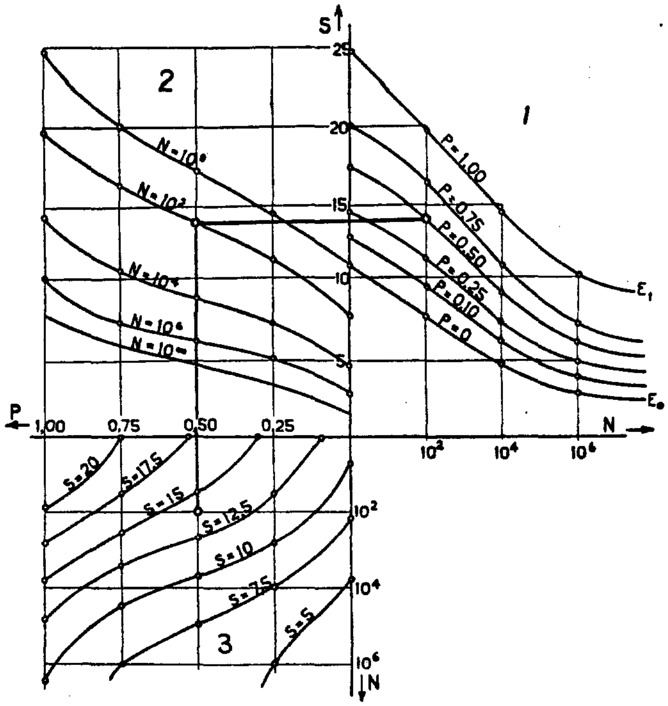

Fig. 4. The complete fatigue diagram.

important one for the designer and sometimes may be all he wants to know. Field 2 gives the $P - S$ curves for some values of N between 1 and ∞ and field 3, the $P - N$ curves for given values of S.

From a theoretical point of view the three representations are equivalent, as it is possible to deduce anyone of them from the other two. From a practical point there is some difference. The quantities U and E cannot be determined directly by experiment, as the former requires an infinite number of tests, and the latter, an infinite number of load cycles. Thus, the only way to obtain them is to extrapolate either by graphical or by analytical methods from values obtained with finite test factors.

In order to determine these values, a number i of loads S_μ are applied to a number of specimens n and their individual lives N_ν as the numbers of load cycles are measured.

As an example, we take 3 different loads ($i = 3$) and 4 specimens ($n = 4$) at each load. The 12 values of N are plotted as load against

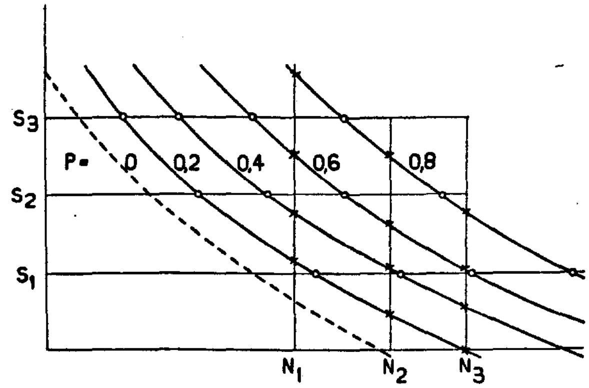

Fig. 5. The $S - N$ curves for different P-values.

number of cycles (The circles in Fig. 5). The only practicable scheme is to test the same number of specimens for each load, because then we have immediately i points on each one of the P-lines. With a different number of tests for each load it is possible to obtain the value of N for a given P only by interpolation, thus introducing an extra error. Strangely, this simple rule is for the most part neglected. As an example, an investigation of RUSSELL and WELCHER (3) may be shown in Fig. 6. Here the 16 values are distributed with 5 at one load, 2 at another load and one at the remaining 9 loads. There is, indeed, a clear impression that the scatter is considerable but it is impossible to have an estimate of the magnitude. Certainly, more information should have been obtained with 4 tests at 4 loads. On the whole, it is not to be recommended to test only one specimen at each load. From the preceding, it may be clear that in this way we may have i very uncertain values for $P = 0.5$ and accordingly one very dubious curve with no information at all about the scatter.

Sometimes, several specimens are tested at each load but only the arithmetic mean of the N-values are given. There are two objections against this praxis. In the first place, the arithmetic mean generally has no definite connection to any P-value. Only if P is a linear function of N the mean coincides with the median. This may be approximately the case at low numbers of cycles, but certainly not at great numbers. If one of the N be infinite, the mean, but not the median, is infinite too. In the second place, by giving the mean only, and not all of the values, no information about the scatter can be obtained, thus throwing away much valuable and

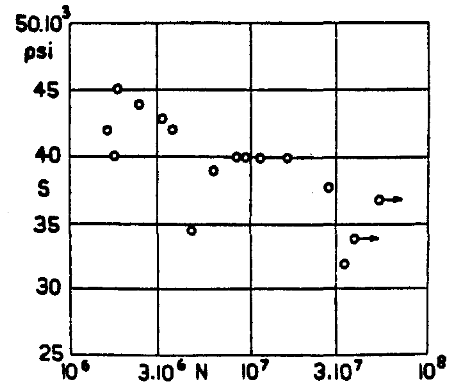

Fig. 6. Endurance test on Monel Metal. RUSSEL and WELCKER (3).

time-absorbed material. This may be exemplified by an investigation of JOHNSSON and OBERG (4). The curves in Fig. 9 for some aluminium alloys are drawn through average points, each point representing the average of several determinations for each stress, except the points above 400 000 000 cycles, which represent the failure of only one or two specimens. For example, the point for a stress of 18 000 psi is 20 390 000 cycles (for alloy C), which represents the average obtained on 12 test specimens. The lowest number of cycles obtained in computing this average was 11 803 400; the greatest number of cycles, 24 931 000. This is the only remark about the quite considerable scatter.

It is obvious that only a part of the $S - N$ field can be explored experimentally, as the numbers of specimens and load cycles are limited. If the tests are stopped after a certain number of cycles, say 10^6 or 10^7, and if this had been N_2 in Fig. 5, the measured values would have been bounded by the five lines S_1, S_3, $P_1 = 0{,}2$, $P_4 = 0.8$ and N_2. Accordingly, failure had occurred in 8 and not in 4 of the tested specimens.

Now, the explored part of the $S - N$ field has to be extended to the ultimate boundaries $P = 0$ and $N = \infty$ by extrapolation. This can be done either graphically or analytically. The former method gives a very inaccurate result; the latter method leaves much better values, provided the proper analytical expressions are used. In this case it is possible to start the procedure, either by letting $N \rightarrow \infty$ or by letting $P \rightarrow 0$. So far, my experience has shown that the first alternative is the better. For this reason we start with a function $N = f(S)$ including an appropriate number of parameters. One of them is E. Generally, the parameters are functions of P only and have to be computed for each P-line by methods which will be demonstrated in the sequel.

This done, the extension $P \rightarrow 0$ may in some cases be carried through satisfactorily with graphical means by extrapolating the parameters, and this if the variation of the parameters is small. Otherwise, we have to calculate for some given values $N_1\, N_2\, N_3$ the corresponding P, S-values (the crosses in Fig. 5) by means of the equations $N = f(S)$. These values may then be used as interpolated observed values to determine the parameters of the distribution functions $P = F(S)$.

When the parameters, among which U is one, are calculated we have obtained the N-curves of field 2 in Fig. 4, which finally may be used to correct the parameters of the P-curves.

This procedure will be demonstrated in detail in the sequel.

3. S—N Curves

The current praxis of running a fatigue test is adequately described in the excellent book, »Prevention of the Failure of Metals under Repeated Stress» (5) as follows: »The endurance limit is determined by preparing a number of specimens of the same material, repeatedly applying stress of known intensity to each, and recording the number of stress cycles each one endured before fracture. By using several specimens, each tested at a lower stress intensity than the preceding one, and hence breaking at a larger number of cycles, a stress is finally found that does not cause failure, no matter how many cycles the patience of the operator allows him to apply. For steels, a life of ten million cycles affords reasonable assurance that the endurance limit has been reached, and except in extremely strong steels, a life of even two million cycles is fair assurance that the endurance limit lies at a stress not far below that which gave such a life».

»The stress, S, is commonly plotted on the ordinary Cartesian scale. The number of cycles, N, may be similarly plotted, but more commonly N is plotted on a logarithmic scale, so that the plot does not take too much space, and so that there is less danger of an

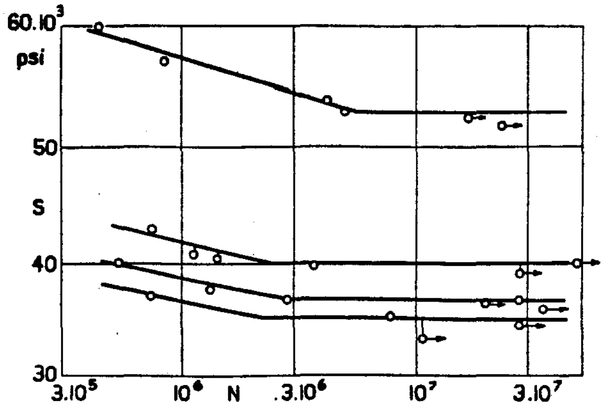

Fig. 7. Endurance tests on shafts of different sizes. PETERSON and WAHL.

inaccurate selection of the endurance limit. This is »semi-log» plotting. If, as is sometimes convenient, stress is also plotted on a logarithmic scale, we have a »log-log» plot. No matter what method of plotting is adopted, the curve for steel changes direction, or shows a »knee», when the stress is slightly above, but closely approaching that for the endurance limit. Around the knee of the curve, a small decrease in stress results in a very large increase in life.»

It is very usual to draw the $S-N$ curve as two straight lines, one sloping and one level, as shown in Fig. 7.

The principal objection against this described praxis is the too small number of tested specimens for each load (very often one or two only). In this way, the accuracy is very small and accidental grouping may give the impression of a knee, without any physical significance. On the contrary, one has drawn the erroneous conclusion that if there is no sharp knee in the $S-N$ curve the material has no definite endurance limit. This not being the case is shown in Fig. 8, taken from an investigation by RAVILLY (6). The $S-N$ curve is a very smooth curve without any knee, but this is not caused by the lack of an endurance limit which is about 6.5 kg/mm². The real reason for the smoothness of the curve lies in the fact that there have been 20 specimens tested for each of 10 loads. With the same number of specimens, RAVILLY has obtained equally smooth curves for six other materials with definite endurance limits.

From these observations it must not be concluded, of course, that all materials have a definite endurance limit. Fig. 9 shows a curve

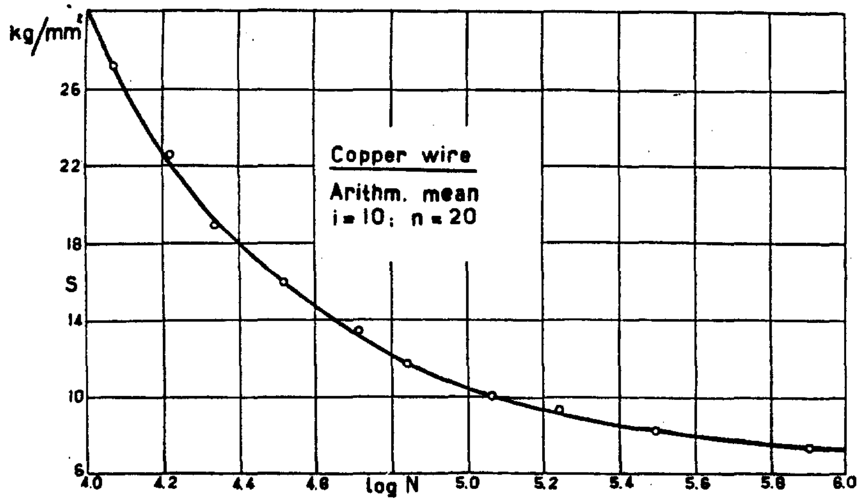

Fig. 8. $S - N$ curve for copper wire. RAVILLY (6).

taken from the previously mentioned paper by JOHNSSON and OBERG (4) about the fatigue resistance of some aluminum alloys. As will be shown in the sequel, the endurance limit = 0 for these materials. The number of specimens is satisfactorily large as each point represents the average of 12 determinations for each stress with some exceptions.

A comparison between Figs. 8 and 9 shows the difficulty of

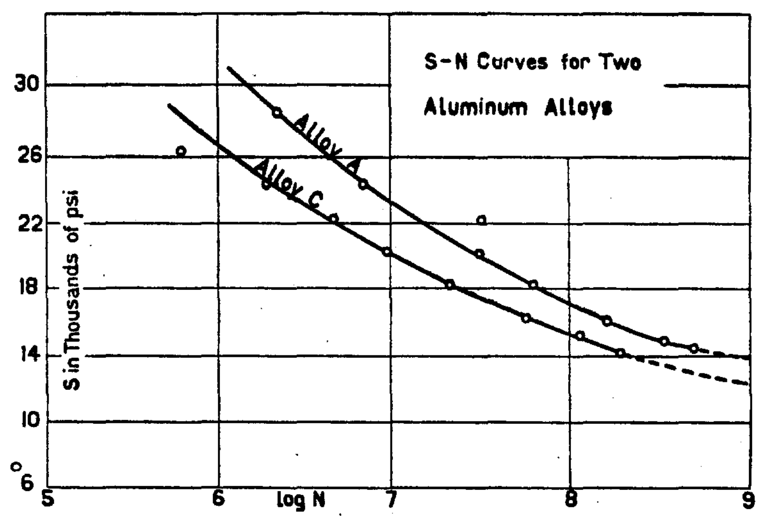

Fig. 9. Rotating-beam test on aluminum. JOHNSON and OBERG (4).

deciding whether or not the material has an endurance limit. This is pointed out by the Authors, who write: »The prolongation of the curve by a dotted line is simply to indicate that it is the opinion of the present investigators that the fatigue limit of the aluminum alloys is something less than indicated by the solid line, but that the point at which the curve becomes asymptotic is undetermined. In fact, there is no conclusive evidence to show that the curve ever becomes asymptotic.»

From the cited papers it is apparent that an analytical relation between S and N is wanted in order to determine the endurance limit with any accuracy.

Among others, FREUDENTHAL (7) has tried to deduce such a relation theoretically from the static distribution function of the ultimate strength in this way: »From the curve $P = F(S)$ the increase in the probability of rupture resulting from repetitions of the load S can be evaluated. If P is the probability that a cohesive bond will be destroyed by action of the load S, $1 - P$ is the probability that it will not be destroyed. The probability that it will not have been destroyed after N repetitions of the load is, according to the rules of the theory of probability, $(1 - P)^N$. Consequently, the probability that the bond will not sustain N repetitions without destruction becomes $1 - (1 - P)^N$, which is larger than P and for constant P increases with N. Hence, the probability of rupture of the individual bond steadily increases with the number of load cycles.»

This deduction is not correct, as the applied rule of probability is valid only on the condition that the events are independent of each other. This is not the case, as some slight change in the specimen must occur after each load cycle. Otherwise the specimen would endure an infinite number of load cycles, if it only could endure the first.

The formula of Freudenthal gives the probability of fracture in a population of N specimens, to which the load is applied without any repetition. Besides, it seems out of the question to deduce the $S - N$ curve from the statical distribution function only, as the progressive change in the structure of a specimen under repeated stresses may be regulated by laws that cannot be completely unveiled by testing a specimen at static loading.

If, for the present, our knowledge is insufficient to develope a rational theory of the relation between load and life, the only remaining way is to choose a plausible analytical function for this relation.

Then, we start from the evident assumption that each specimen has its individual endurance limit E. If a load $S \leq E$ is applied, the life will be infinite, but if the load is greater, the life will be finite. The greater the difference $S - E$, the smaller the life N.

It should be well observed that, at least for the moment, we have no means of measuring the endurance limit of an individual specimen If we, by way of trial, apply a load to the specimen in question, and find it capable of carrying this load, even then we could not repeat the experiment, as the endurance limit has been changed by the first trial. It is, however, possible to determine experimentally the distribution function $F(E)$ but only approximately. Even then it is a very time-absorbing procedure.

For the moment we have to be satisfied with the statement that the distribution function $F(E)$ exists. Furthermore, we make the plausible assumption that an individual with a greater endurance limit E has a greater life N at every load S. This is not at all self-evident, but it is the same as the assumption that there are no intersections between the P-lines in field 1 of Fig. 4. From this it follows that if the values at a given load are arranged in the order of increasing magnitude of N, they will also be arranged in the order of increasing magnitude of E.

Accordingly, if we select all values with the same P, one from each load, we may assume that all of them have the same E-value, within the sampling error. If the number n is the same for every row, ($n = 20$ in tab. 1) then all values with the same ν, as $P = \frac{\nu}{n+1}$, i. e. each column, have the same E.

We now put, and this has so far proved very satisfactory,

$$N = k\,(S - E)^{-m} \qquad (15)$$

or

$$\log N = \log k - m \log (S - E) \qquad (16)$$

If we denote

$$y = \log N;\ a = \log k;\ x = \log (S - E) \qquad (17)$$

then

$$y = a - mx \qquad (18)$$

TABLE 1 a. *Number of thousands of stress cycles N of 1.5 mm copper wires in reversed torsion.* RAVILLY (6).

S kg/mm²	μ \ ν	1	2	3	4	5	6	7	8	9	10
7.25	1	423.0	439.0	441.0	509.0	571.0	616.0	648.0	664.0	685.0	689.0
8.25	2	172.0	226.0	229.0	253.0	286.0	290.0	303.0	312.0	318.0	325.0
9.25	3	104.0	110.0	137.0	149.0	151.0	163.0	164.0	166.0	171.0	174.0
10.0	4	72.0	83.0	93.0	95.0	102.0	103.5	105.0	108.0	112.5	113.0
11.75	5	44.5	45.0	46.1	57.2	62.1	62.5	64.0	67.7	68.2	69.6
13.5	6	31.0	38.0	43.3	43.6	44.8	44.9	46.0	46.5	48.3	49.0
16.0	7	18.0	21.2	24.0	25.2	26.0	26.2	28.1	29.7	32.4	32.8
19.0	8	15.0	16.6	16.6	16.7	18.5	18.6	19.0	19.9	20.2	20.5
22.5	9	8.8	9.8	11.6	12.0	12.2	13.2	14.4	15.0	15.1	16.9
27.0	10	4.8	6.0	7.6	8.7	9.0	9.0	9.8	10.2	10.5	10.8

TABLE 1 b. *Number of thousands of stress cycles N of 1.5 mm copper wires in reversed torsion.* RAVILLY (6).

S kg/mm²	μ \ ν	11	12	13	14	15	16	17	18	19	20
7.25	1	767.0	842.0	886.0	910.0	978.0	1 117.0	1 135.0	1 137.0	1 192.0	1 311.0
8.25	2	328.0	332.2	340.0	342.0	343.0	347.0	358.0	373.0	400.0	431.0
9.25	3	179.0	184.0	189.0	190.0	208.0	208.0	214.0	216.0	224.0	238.0
10.0	4	115.5	117.0	121.5	126.0	127.5	128.0	130.0	131.0	137.0	151.0
11.75	5	72.0	73.0	74.5	74.8	80.5	82.0	82.0	84.5	87.8	96.7
13.5	6	53.8	54.0	56.0	57.9	59.9	60.3	60.5	61.0	63.8	67.7
16.0	7	34.0	34.1	34.3	35.1	38.8	39.0	43.0	46.0	46.1	46.4
19.0	8	21.3	21.8	22.0	24.3	24.5	25.5	25.9	26.3	30.2	30.6
22.5	9	17.4	17.8	18.0	18.0	18.7	21.7	21.9	22.4	22.7	24.0
27.0	10	12.2	12.4	12.4	13.6	13.9	14.0	14.4	15.3	16.1	18.8

Here, the three parameters k, m, and E, are functions of P only, i. e. they are constant for each given P but have, in general, not the same values for different columns of the table.

These equations will now be applied to the results of a very careful and extensive investigation by RAVILLY (6), who has tested wires from several materials at reversed torsion. Most of the tests have been carried through with twenty specimens ($n = 20$) at ten loads ($i = 10$). It should be noted that the tests have been run with constant amplitude, i. e. constant maximum strains. This method certainly gives somewhat higher N-values than with constant maximum stresses, which have been calculated from the amplitude and the initial modulus of rigidity.

As the first example, we take the values for *zinc*. It is to be regretted that not the twenty N-values themselves, but only their arithmetic means for each one of the ten loads are given. We have to consider them as substitutes for the means ($P = 0.5$) and determine the three parameters of Equ. (15). From (18) it follows that y is a linear function of x as a and m are constant for a given P. If we then plot the points (x, y) on a Cartesian scale, they will fall on a straight line, provided the exact E is selected, otherwise on a bent curve. This property will be used as a criterion on the correctness of E. A graphical method will do if the scatter is small. If not, numerical computation is unavoidable, as will be shown in the sequel. In each case, it is recommendable to start with a graph in order to ascertain whether the material is statistically homogeneous. Sometimes it has proved necessary to split up the values in two or more parts.

A suitable praxis is to start with $E = 0$. Using a log-log plot, the values of table 2 fall on a bent curve, as shown in Fig. 10. This proves that $E \neq 0$. Now, we select some plausible values of E, for instance 5.0, 5.5, and 6.0, and plot the values on the log $(S - E)$ scale. With some experience, it is possible to find, after one or two trials, the correct value; in this case, 5.5. (One of the values

TABLE 2. *Reversed torsion endurance test of zinc wires.* RAVILLY (6).

S kg/mm^2	log S	log $(S - 5.5)$	N . 10^{-3}	log N
6.50	.813	.000	360.0	5.556
6.75	.829	.087	270.0	5.431
7.25	.860	.243	190.0	5.279
8.50	.929	.477	150.0	5.176
8.75	.942	.512	96.0	4.982
12.00	1.079	.813	42.0	4.623
17.25	1.237	1.070	22.5	4.352
19.50	1.290	1.146	18.0	4.255
21.00	1.322	1.190	14.4	4.158
24.25	1.385	1.273	12.3	4.090

Note. The values of N are the averages of 20 determinations for each load.

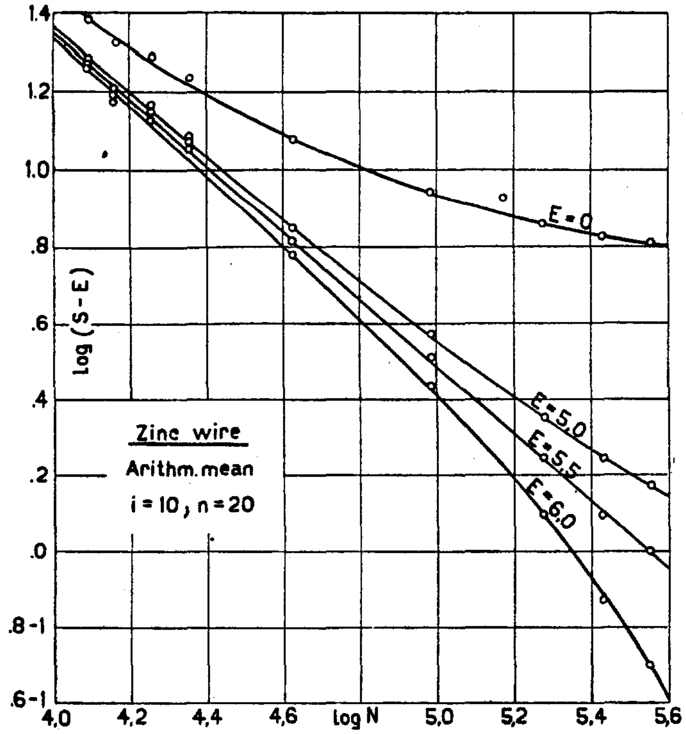

Fig. 10. Reversed-torsion test on zinc. Ravilly (6).

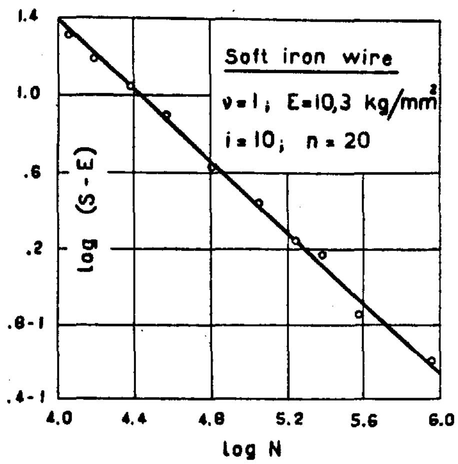

Fig. 11. Reversed-torsion test on soft iron (Armco). Ravilly (6).

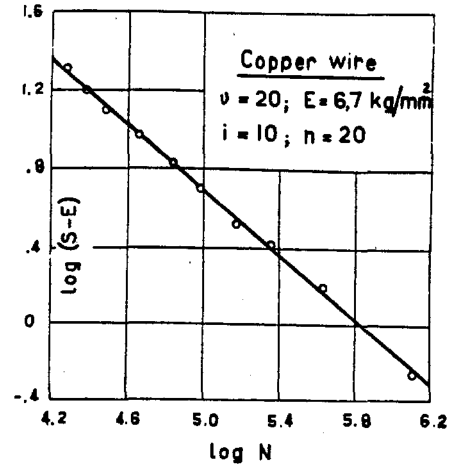

Fig. 12. Reversed-torsion test on copper. RAVILLY (6).

obviously falls out of bounds and is therefore excluded after the first plotting).

For *soft iron* (Armco 99.8 % Fe) all of the 200 values are given. Fig. 11 shows the curve for $\nu = 1$ and $E = 10.3$ kg/mm². The curves for other ν are straight parallel lines with the same E-values, from which it follows that the only difference between the curves lies in

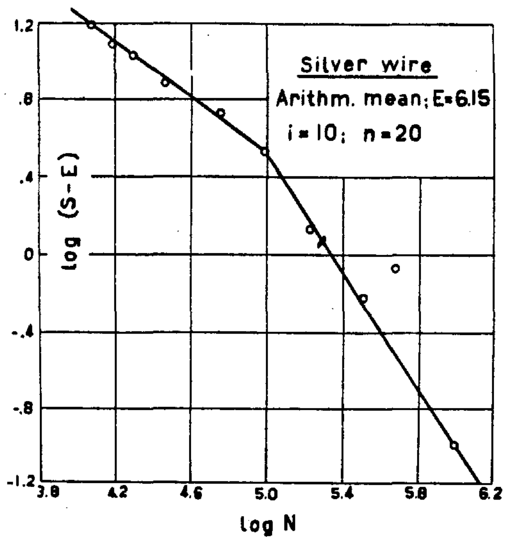

Fig. 13. Reversed-torsion test on silver. RAVILLY (6).

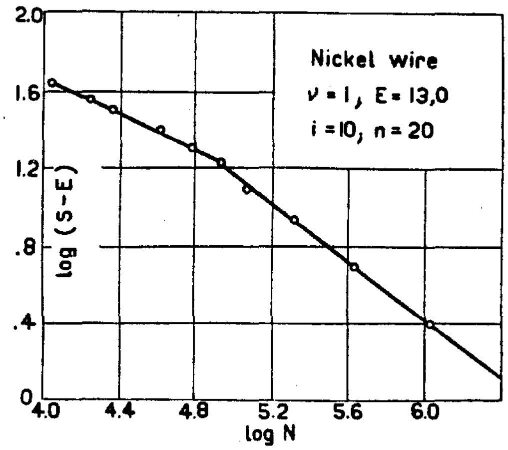

Fig. 14. Reversed-torsion test on nickel. Ravilly (6).

the parameter k. This is an unusual case, and may be caused by the fact that the specimens are selected, with the aid of a magnetic method, which has considerably reduced the scatter.

Fig. 12 shows the curve for *annealed copper* and $\nu = 20$.

I think it is not too much to say that the $S - N$ curves for these three materials are reproduced in an excellent way by Equ. (15).

As mentioned before, it does not always happen that the $S - N$ curve is one single straight line on the log-log plot. The values of Ravilly for *silver* are given in Fig. 13. The arithmetic means of 20 N-values are plotted (One of the values is decidedly out of bounds).

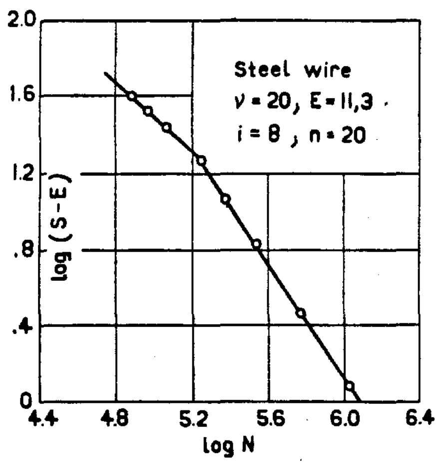

Fig. 15. Reversed-torsion test on steel. Ravilly (6).

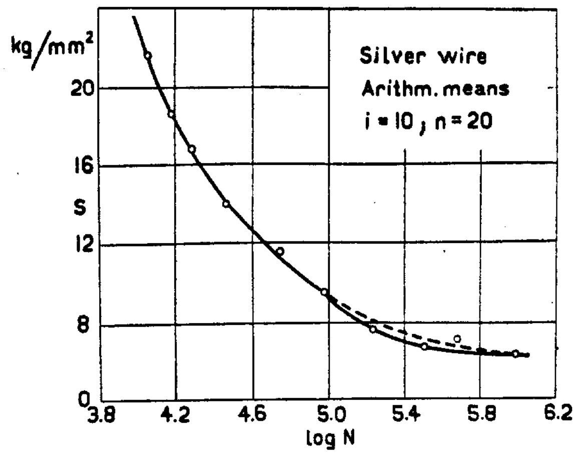

Fig. 16. $S - N$ curve for silver. RAVILLY (6).

The curve consists of two straight lines, with the discontinuity at $N = 10^5$.

The same type of curve is shown in Fig. 14 for *annealed nickel*, and in Fig. 15 for *annealed steel*. This seems to be the normal type. Probably there exists such a discontinuity also in the previous curves for zinc, iron, and copper at higher loads. At any rate, the extrapolated values for $N = 1$ exceed considerably the ultimate static strength of the materials.

For the moment, our knowledge is too small to tell if and how the point or points of discontinuity are connected with some static properties, such as the yield or elastic points of the material. I think it would be of great importance if such a connection could be established.

Before closing this chapter it seems worth while to call attention to the advantage of using the log $(S - E)$-log N plot instead of the usual semi-log, (S-log N) plot.

It is evident that for several reasons a straight line is to be preferred to a bent curve. It facilitates the computation of parameters, accidental deviations of observed values are more easily detected, and so are discontinuity points; it simplifies inter- and extrapolation etc.

This will be exemplified by reproducing the lines of Fig. 13 on a semi-log plot as shown in Fig. 16. I think, nobody would without knowing the representation of Fig. 13, have drawn the solid line

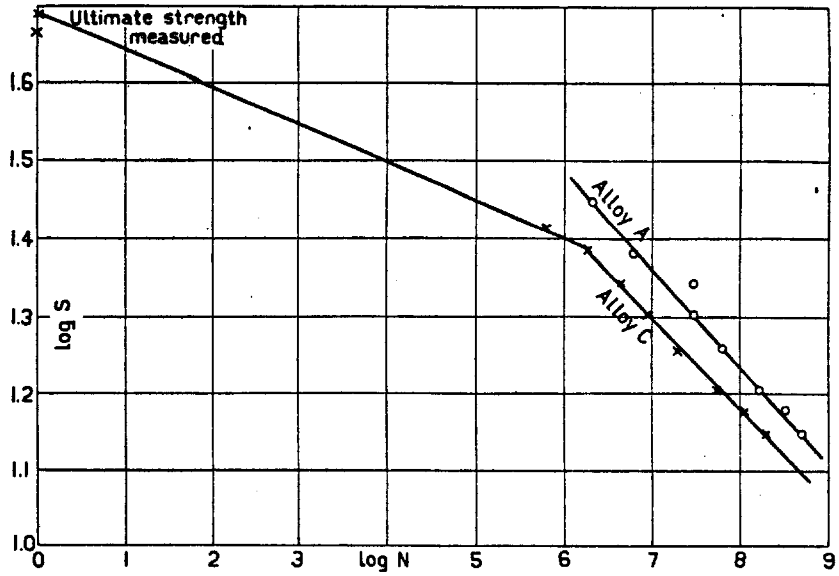

Fig. 17. Rotating-beam test on aluminum alloys. JOHNSON and OBERG (4).

in Fig. 16, which doubtless is the correct one, but instead, the dotted line. He would probably estimate the three last values but one to be impaired by about equally great errors. From Fig. 13 it is easily seen that the last value but one (log $N = 5.68$) is much more erroneous than the two others. The discontinuity point is hard to detect in Fig. 16, but not in Fig. 13.

Furthermore, as already mentioned, Johnsson and Oberg find no conclusive evidence to show that the curves in Fig. 9 ever become asymptotic, but when the values are plotted as in Fig. 17 it seems obvious that $E = 0$ for the two aluminum alloys, as the points fall quite closely on a straight line, with two exceptions. One of them (Alloy A) is with all probability affected by a great accidental error and may be excluded. The other one is the smallest N-value of alloy C. Its deviation from the straight line would probably have been estimated as an accidental error, had not the discontinuities of the previous curves been known. Now, if we draw a line through the two lowest N-values, this line goes almost exactly for $N = 1$ through the value of the ultimate static strength of the material, as given by the investigators. If this observation is correct, there would be a discontinuity at log $N = 6.3$ and the $S - N$ curve would be represented by two straight lines from $N = 1$ to ∞. This conclusion has to be taken with some reservation as the observations, of course, are very few in this regard, being only two points, representing the average of 12 determinations.