ANALYSIS OF FATIGUE TEST RESULTS

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The objective of a fatigue test is, from a designer's point of view, to determine the fatigue life of a test piece, a specimen, or a component, subjected to prescribed sequences of stress cycles so as to predict the fatigue life in actual service.

In early days, the main interest was concentrated on that load below which no fatigue failure would occur, because at that time the designer could practically always reduce the stresses by increasing the dimensions of his components sufficiently to stay within the critical stress, or fatigue limit. In other words, machine parts could be designed for unlimited life.

This happy situation does not exist any more, at least not with regard to flight structures. Consequently, knowledge of the relationship between load and life has become more important.

The normal method of carrying out a test for this purpose was to place a specimen in the testing machine and to apply reversals of stress of a certain constant amplitude.

If the specimen failed after a certain number of cycles, a second specimen was placed in the machine and subjected to a range of stress less than that used for the first specimen. Again failure resulted after a certain, but longer endurance period, and the procedure was repeated, the applied range of stress being reduced with each succeeding specimen, that is, with one single specimen for each stress level, until a stress was reached which did not cause failure. The relationship between the applied load S and the fatigue life N, measured in number of stress cycles, was usually considered to be composed of two straight lines as indicated in Fig. 1.

At this stage in the history of fatigue testing, the scatter in fatigue life was not recognized as a problem. Even in a fairly recent manual I have found the statement that in fatigue testing it is neither necessary nor desirable to use many specimens.

During the last decade it has become a generally accepted view that scatter in fatigue life is an inherent property of the material and, thus, that fatigue life is a stochastic quantity, that is, a quantity that cannot be specified by a single number but has to be specified by a function called the cumulative distribution function, giving the probability that the quantity takes a value equal to or less than the argument of the function, or, mathematically expressed

$$Cdf(X) = Prob\ (X \leq x) = F(x) \qquad (1)$$

The general appearance of such a distribution function is presented in Fig. 2. The function $F(x) = 0$ for a value $x = x_u$ = the lower limit of the distribution, which is not necessarily equal to zero as is sometimes assumed, is an increasing, or at least never decreasing, function of x, that is, $F(x) = 1$ (as being a probability) for $x = \infty$.

It is a well-known fact that the experimental determination of such a function requires a large number of tests, particularly when very small probabilities are desired.

As long as the scatter in fatigue life was neglected it was considered sufficient to know the relationship between load and life, the S-N relationship. Now that it has become clear that a statistical approach is indispensable, we have to ask for a P-S-N relationship, that is, a relationship between the probability of failure P, load S, and life N, and also how this relationship can, in the most efficient manner, be determined from actual fatigue testing.

The only way of representing the relation between three quantities by use of a two-dimensional graph is to draw a set of curves linking two of the quantities, each curve corresponding to selected values of the third quantity. This can, of course, be done in three different ways, but the most common and easiest way to represent actual fatigue tests is to draw a set of S-N curves, each for a given probability P, as demonstrated in Fig. 3. One of the curves, the median curve, corresponding to $P = 0.5$, should be used as the average S-N curve.

Since the complete diagram covers the field from $N = 1$ to $N = \infty$, or $N = 10^8$, say, as in many, if not all, cases is a good approximation of infinity, it is convenient to use the semi-logarithmic scales S and log N. If we define N as the number of cycles during which failure occurs, $\log N = 0$ corresponds to the static test, since failure does occur during the first cycle. (Some people prefer to say $N = 1/4$, but if we use an S-N equation involving $N + B$, where B is a large parameter, no decision between the two alternatives is required.)

Sometimes the average curve has been determined by taking the arithmetic mean of the observed life-times at each stress level, log N, and to draw a curve through these points instead of through $\log \breve{N}$, where $\breve{N}$ is the median. This procedure should be avoided, even though log N has less scatter (variance) than $\log \breve{N}$, because such a curve does not correspond, in general, to a fixed value of the probability P and thus may intersect the actual P curves. It should be noted that the diagram in Fig. 3 can be applied to different types of loading. In the case that each stress level corresponds to repeated, constant stress cycles which can be defined by two values, for instance, $S_{max} \div S_{min}$; $S_a \div S_m$; $S_{max} \div R$, etc., S may represent S_{max} for constant S_{min} or R, or S_a for constant S_m, or even S_m for constant S_a. But the diagram is applicable also to program tests and random-load tests, provided S represents an appropriately defined quantity, such that a higher value corresponds to a shorter life.

The diagram in Fig. 3 can be cut in vertical or horizontal directions, and points on these lines can be experimentally determined. The only vertical line on which easily determined points will fall is that which corresponds to $N = 1$ or $\log N = 0$. The ultimate tensile strength may be considered a fatigue strength, corresponding to $N = 1$, just as any fatigue strength has to correspond to a specified value of N.

The horizontal cuts are easily obtained by tests at fixed stress levels and by recording the fatigue life of each specimen. Vertical cuts for values of N other than $N = 1$ require tests at several different stress levels and many more tests than the above-mentioned two types of cuts, because the cycle life aimed at will never be exactly attained. Nevertheless, the fatigue strength is as good a stochastic quantity as the fatigue life, and it is, in some respects, of more importance than the fatigue life. I will presently demonstrate that its distribution function can be determined experimentally, without any assumption with regard to the load-life relation, and with any desired accuracy, provided a sufficiently large number of tests are carried out.

For the further discussions, the concept of individual S-N curves will be introduced by postulating that there exists for every individual specimen such a curve that involves parameters, which take values varying from specimen to specimen. It is further assumed that in every point of this curve

$$ds/dN < 0 \tag{2}$$

that is, S is a decreasing function of N.

This assumption can never be verified by experiment, because we cannot measure more than one single point of the curve, but it is almost self-evident, since it implies that a higher load corresponds to a shorter life and vice versa. Let us now suppose that it is required to determine the fatigue strength for a given cycle life N_j. For this purpose n specimens are tested at the stress level S_i with the result that m specimens failed at a shorter life than N_j and (n-m) specimens survived this number of cycles.

It can be proved that the ratio $m/(n-m)$ is an unbiased estimate of the proportion P of the population that has a fatigue life less than N_j. Consequently

$$P = m/(n-m) \tag{3}$$

The accuracy of this estimate can be measured by its variance

$$\sigma_P = P(1-P)/(n+2) = m(n+1-m)/(n+1)^2(n+2) \tag{4}$$

From assumption (2) it immediately follows that all specimens that failed prior to N_j, namely m specimens out of n, will have a fatigue strength less than S_i, while all specimens that survived N_j cycles, namely $(n-m)$ specimens, will have a fatigue strength larger than S_i as indicated in Fig. 4. The point (S_i, N_j) and any other such point consequently is a common percentage point of the life distribution and of the strength distribution. It should be noted that the n observed fatigue lives provide n percentage points of one life distribution and one percentage point of n strength distributions.

If this procedure is repeated for different stress levels, a number of points of the strength distribution is obtained. Such a test with several stress levels in order to determine the Cdf(S) is called a probit test and its primary purpose is to determine the distribution of the fatigue limit. The same method of analysis can, of course, be used to examine the strength distribution at any pre-assigned cycle of life.

Note that the percentage points of the strength distribution thus obtained are independent of each other, while those of the life distribution (as being order statistics) are not. This difference implies that an efficient estimation of the strength distribution parameters requires knowledge only of the variances of the data points, as given by Eq. (4), whereas that of the life distribution requires also the covariances. This fact makes it much easier to determine, from the observed values, the distribution of the strength than that of the life, which may serve as a compensation for the more difficult experimental determination of the data points of the strength distribution.

From the preceding considerations it may be concluded that the numerical evaluation of data from a test series that has been carried out in order to provide a diagram as indicated in Fig. 3, including several stress levels and possibly some static tests and probit tests, will require three analytical expressions, viz., (a) for the average S-N curve (P = 0), (b) for the life distribution, and (c) for the strength distribution.

Obviously, the P-S-N diagram can be established if the average curve and one of the distributions are known for a sufficiently large set of points.

1. <u>The Average S-N Equation</u>

An equation which has successfully been applied to several hundred test series, some of them very large, can be put in the following form:

$$S = (S_u - S_e)(N/B + 1)^{-a} + S_e \qquad (5)$$

It involved four parameters, S_u, S_e, a and B, where

S_u = median of the static strength

S_e = median of the fatigue life

a = shape parameter

B = scale parameter of time. It has the dimension of the cycle life N and is the rational unit of fatigue life. It has been found that the value of B is considerably smaller for notched specimens and components than for plain, unnotched specimens.

These four parameters can be determined from the observed data points by graphical analysis or by analytical evaluation.

The graphical method is less precise, but it is recommended to start the analysis by this method because it gives an immediate and clear picture of the S-N relationship.

If the observations cover a large part of the S-N field and sufficient precaution has been taken, for instance, with regard to speed of the testing machine, efficient cooling of the specimen, etc., then the following method can be used.

The median data points are plotted on semi-logarithmic scales and the average S-N curve $(P = 0.5)$ is fitted by eye, thereby determining the values S_u and S_e as indicated in Fig. 5.

The maximum (negative) slope of the $(S - \log N)$ curve, defined by

$$d^2S/(d \log N)^2 = 0 \qquad (6)$$

and located at the inflection point (S_i, N_i) is

$$dS/d \log N = -2.3026 \ [a/(1+a)]^{1+a)} \cdot (S_u - S_e) \qquad (7)$$

From this equation it follows that the quantity, in Fig. 5 marked by $\Delta \log N$, is uniquely determined by the parameter a. The distance $\Delta \log N$ is easy to determine, since the slope of the curve is not very sensitive to errors in judging the exact location of the inflection point.

We have

$$\Delta \log N = 0.43429 \ [(1+a)/a]^{1+a} \qquad (8)$$

$$(S_B - S_e)/(S_u - S_e) = 2^{-a} \qquad (9)$$

$$(S_i - S_e)/(S_u - S_e) = [a/(1 + a)]^a \qquad (10)$$

The relationship between the various quantities is given in the following table.

a	log N	$(S_B-S_e)/(S_u-S_e)$	$(S_i-S_e)/(S_u-S_e)$
0.0	∞	1.000	1.000
0.1	6.072	0.933	0.787
0.2	3.730	0.870	0.699
0.3	2.922	0.812	0.644
0.4	2.509	0.758	0.606
0.6	2.086	0.660	0.555
0.8	1.869	0.574	0.523
1.0	1.737	0.500	0.500
1.5	1.557	0.354	0.464
2.0	1.460	0.250	0.444

From the graphically obtained value of Δ log N the corresponding values of a, $(S_B-S_e)/(S_u-S_e)$, and $(S_i-S_e)/(S_u-S_e)$ are read from the table. Since S_u and S_e have been determined, the values of S_B and S_i are computed and the corresponding values of log B and log N_i determined as indicated in Fig. 5.

The preceding method requires stress levels providing both low-cycle and high-cycle values and preferably also static test values. If only a part of the S-N field is observed, a preliminary estimation of the parameters can be made by "rectifying" the S-N curve, that is, by plotting the observations on scales chosen in such a way that the curve becomes a straight line. This can be done in the following way.

Introducing the quantity

$$U = (N/B + 1)^{-a} \tag{11}$$

into Eq. (5) we have

$$S = (S_u - S_e)\cdot U + S_e \tag{12}$$

If we now have found, by trial and error, the proper values of a and B and if we plot the values of S against U, then the median data points will fall (with some deviations due to sampling errors) on a straight line as indicated in Fig. 6. The merit of this plot is that it includes the point $(S,N) = (S_e, \infty)$.

More accurate values of the four parameters can be obtained by applying the Method of Least Squares to this evaluation problem. Since the data points are independent of each other, and since it has been found that in most cases the variance of S is practically independent of N, this method is efficient without any weighting of the observations.

Putting

$$b = S_u - S_e \tag{13}$$

we have

$$S = b\cdot U + S_e \tag{14}$$

With the notations

$$s = S - \bar{S} \quad \text{and} \quad u = U - \bar{U} \tag{15}$$

where

$$\bar{S} = \sum_{1}^{u} S_i/n \quad \text{and} \quad \bar{U} = \sum_{1}^{n} U_i/n \tag{16}$$

and S_i, U_i includes all values, not only median ones, the least sum of squared deviations from the fitted line is obtained by

$$b = \Sigma s.u/\Sigma u^2 \quad \text{and} \quad S_e = \bar{S} - b.\bar{U} \tag{17}$$

The value of S_u is finally computed from Eq. (13).

The variance of S_i is given by

$$\sigma^2 = [\Sigma\, s^2 - (\Sigma\, su)^2]/(n-2).\Sigma\, u^2 \tag{18}$$

The value of σ is a measure of the quality of fit. The smaller it is, the better the fit.

The practical application of these formulae starts with an arbitrary pair (a,B) from which the values of s and u are computed and then the values of b, S_e, S_u and σ^2 from Eqs. (13) to (18).

Now, keeping B constant and varying <u>a</u> we can find a value of <u>a</u> that gives the least value of σ^2 for the given value of B. We now take another value of B and repeat the procedure until we have found a pair (a,B) that gives the absolute minimum of σ^2 compatible with the observations.

When the number of observations is large, this procedure is rather time-absorbing and tedious, and the use of electronic computers is indispensible. Under a contract with the U. S. Air Force through its European office, the Office of Aerospace Research, Brussels, Belgium, these computations have been programmed. Since the observed data points (S_i, N_i) have been fed into the computer, the values of n, B, a, S_u, S_e and σ are computed and printed. The computing time is 1 to 2 minutes for 30 to 40 data points. This program has been applied to several hundred test series under the contract.

Considering the assumption that the distribution of S is practically independent of N, it is legitimate to use the means of S instead of the medians. However, the fitted curve does not necessarily correspond to $P = 0.5$, but to a value of P that is easily obtained by counting all data points lying on and below the fitted curve and dividing this number by the total number of data points.

The next step in the evaluation procedure consists in a determination of the distribution of either the fatigue life or the fatigue strength. The choice between them depends on the observations, but in both cases the same distribution function (with differing parameters) can be used.

The distribution function and estimation methods of its parameters will now be indicated.

2. The Distribution of Fatigue Life

The following distribution has been found to fit the observations quite satisfactorily

$$P = F(x) = 1 - \exp \{-[(x-x_u)/x_o]^m\} \qquad (19)$$

This distribution involves three parameters: the location parameter x_u (also the lower bound of the distribution below which x does not exist), the scale parameter x_o, and the shape parameter m. It can be put in the following form:

$$\log \log [1/(1-P)] = m.\log (x-x_u) - m.\log x_o \qquad (20)$$

If we now have a sample of size n and order the observations from least to greatest, then

$$P_i = i/(n+1) \qquad (i = 1,2,\ldots,n) \qquad (21)$$

Introducing (21) into (22)

$$\log \log [(n+1)/(n+1-i)] = m.\log (x_i-x_u) - m.\log x_o \qquad (22)$$

This equation can be used for graphical estimation of the parameters. Plotting $\log \log [(n+1)/(n+1-i)]$ against $\log (x_i-x_u)$ for an arbitrary value of x_u, the curve will concave downwards, if a too small value has been chosen; the curve will concave upwards if the value is too large. After some trials a value for x_u will be found which rectifies the curve as demonstrated in Fig. 7. The slope of the straight line is equal to m and the intersection with the vertical axis gives the value of $-m.\log x_o$, from which x_o is computed. More accurate estimations can be obtained by analytical methods, for instance, the Maximum Likelihood Method or the Best Linear Unbiased Method of Estimation.

If we now let x in Eq. (19) signify $\log N$ and apply this method to each stress level having a sufficiently large number of tests, say ten or more, then the life distribution for each stress level will result. In a few cases it has been observed that the scale and shape parameters of the individual stress levels do not significantly differ from each other. When this situation occurs, the deviations from the fitted average curve can be pooled into one population, the parameters of which can then be estimated with improved accuracy. Any deviation $\Delta \log N$ from the average curve corresponds to a given value of P. Consequently, the S-N curves for a selected set of P-values can be easily drawn.

3. The Distribution of Fatigue Strength

If we now, on the other hand, let x in Eq. (19) signify the quantity U, defined by Eq. (11) where a and B take the computed values, then we can in the same way obtain the distribution U for each stress level. Since U and S are linearly related by Eq. (14), the distributions of these two quantities have the same shape parameter m, and the location and scale parameters of both distributions are related by Eq. (14).

It has been found that in many cases the scale and shape parameters of these distributions do not depend on the stress levels. If so, the deviations of S from the fitted average S-N curve can be pooled to form one population. Any deviation S from the average curve corresponds to a given probability P, and the S-N curves for a selected set of P-values can be drawn.

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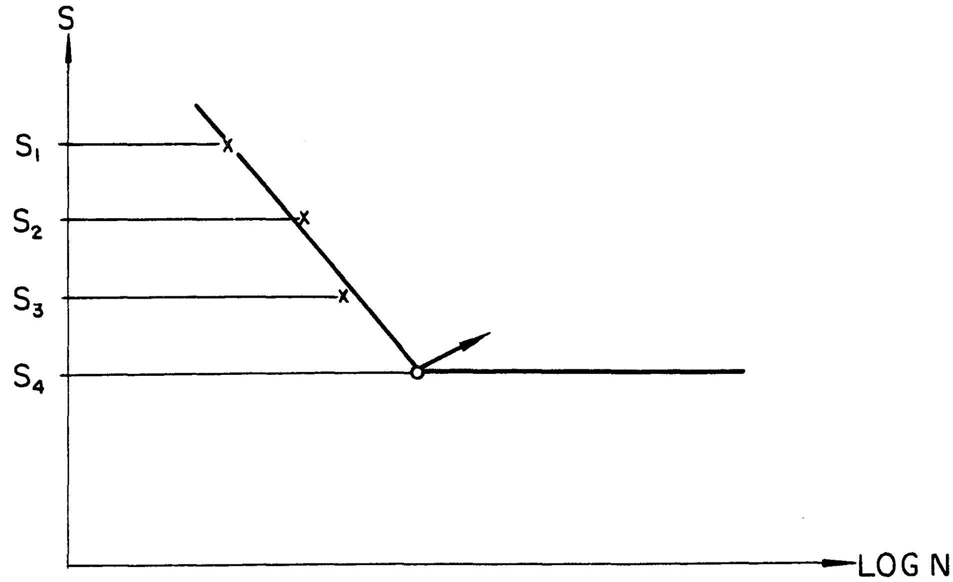

FIG. 1 SCHEMATIC S–N CURVE

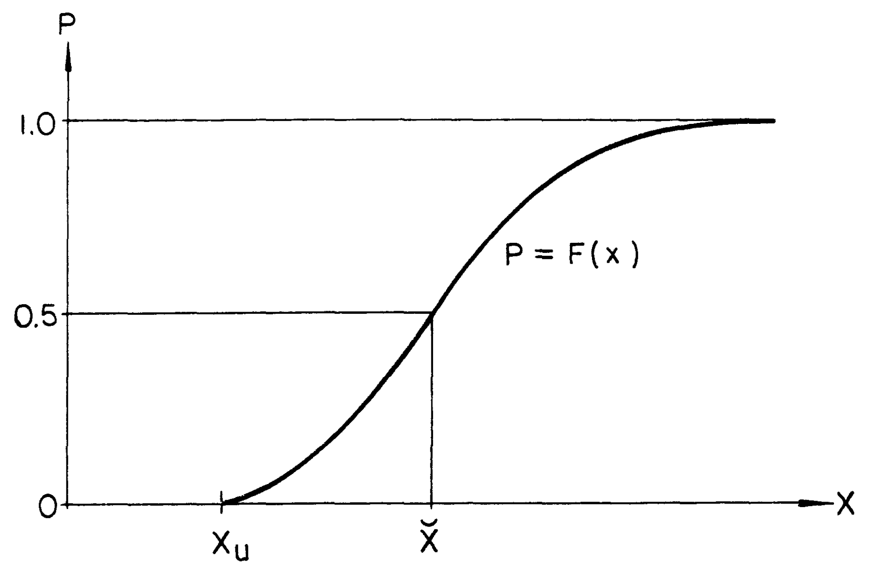

FIG. 2 A CURVE REPRESENTING THE CUMULATIVE DISTRIBUTION

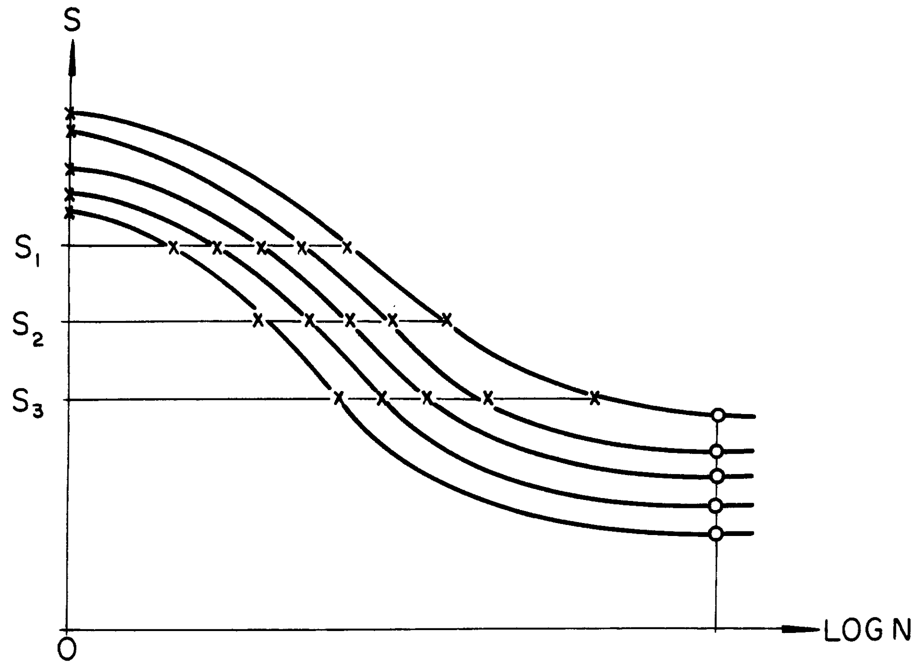

FIG. 3 SET OF S-N CURVES; SEMI-LOGARITHMIC PLOTTING

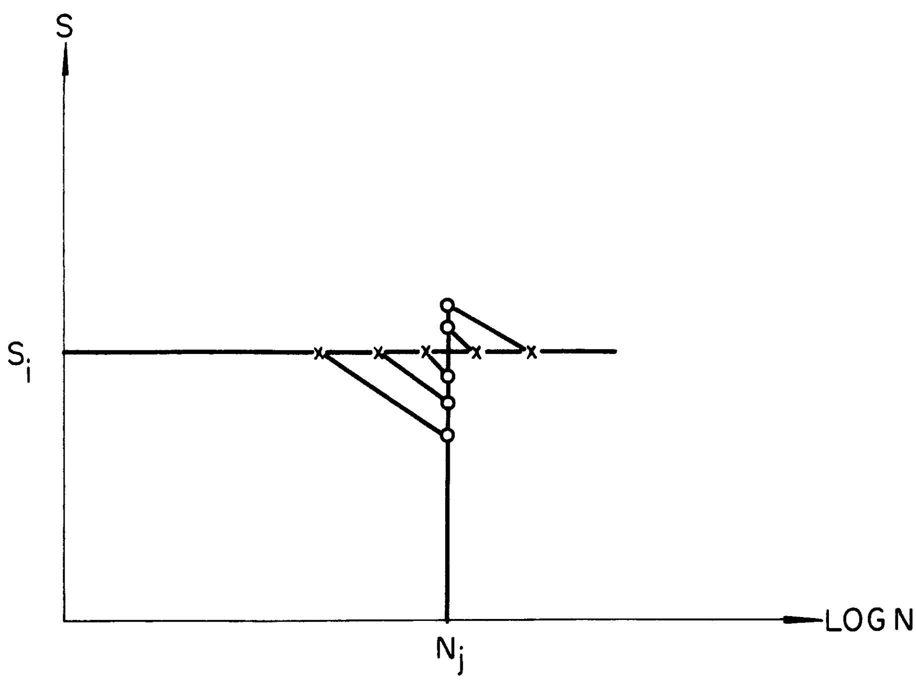

FIG. 4 TRANSFORMATION OF A FATIGUE LIFE DISTRIBUTION INTO A FATIGUE STRENGTH DISTRIBUTION

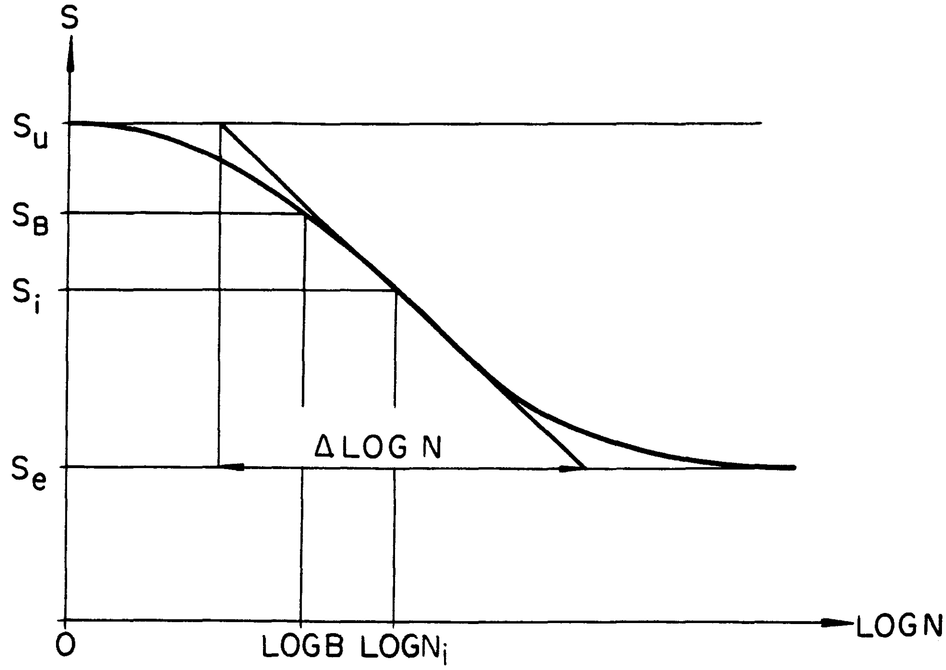

FIG. 5 GRAPHICAL ESTIMATION OF THE PARAMETERS OF AN S—N CURVE

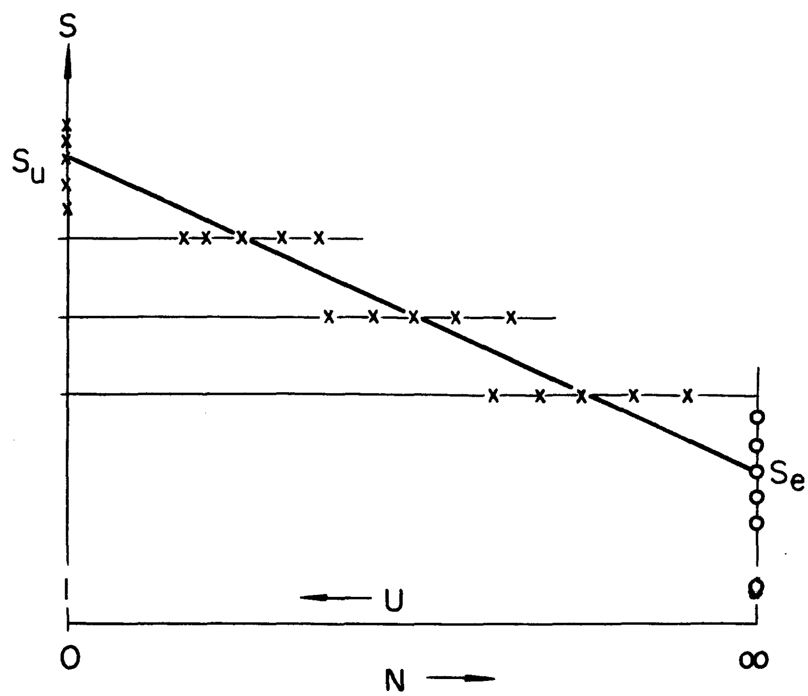

FIG. 6 PLOTTING OF S AGAINST $U=(N/B+1)^{-a}$

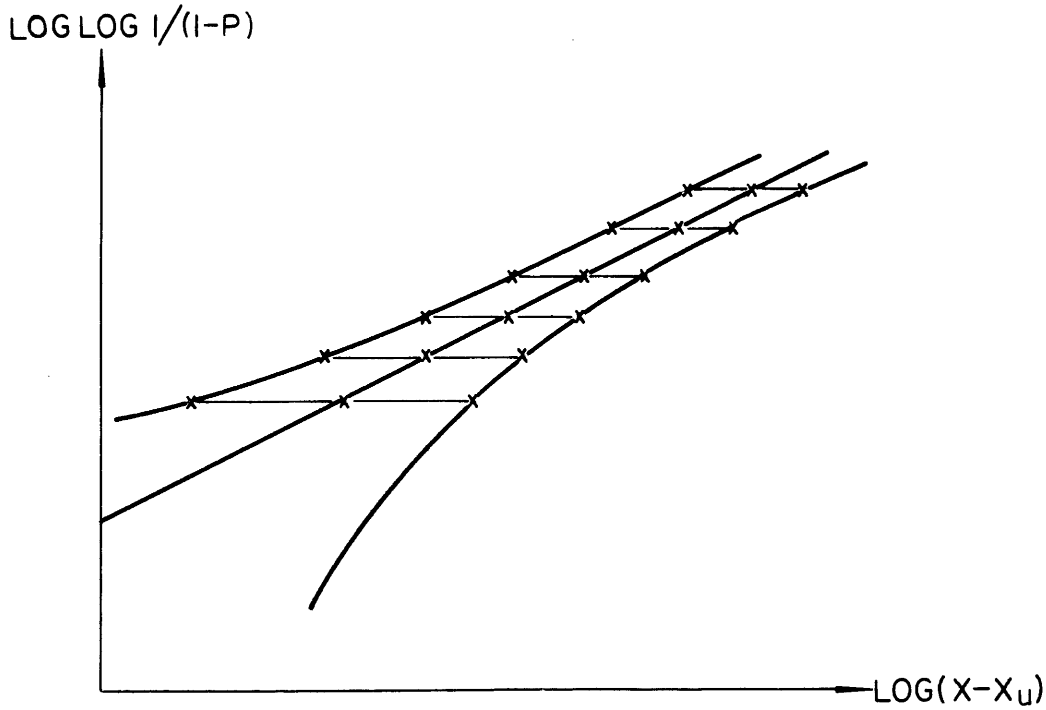

FIG.7 GRAPHICAL ESTIMATION OF THE PARAMETERS OF A FATIGUE LIFE DISTRIBUTION