FATIGUE DESIGN AND RELIABILITY

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1. Introduction

The requirement that modern design should permit a rational estimate of the probability of failure or reliability associated with a specific safety factor renders the conventional, non-probabilistic concept of structural design by criteria based on purely empirical safety factors or load factors (with the tacit implication that it is possible to design a structure with zero probability of failure) obsolete; in fact it is this requirement which finally puts the safety analysis on an equal basis with load analysis and stress analysis. The emphasis is on the word "rational," and the question arises how "rational" a reliability or safety analysis can be made under the conditions encountered in the design, not of small easily tested electronic parts, but of large structures.

It is important to note, however, that already the introduction of the concept of a probabilistic safety analysis represents a significant step in the direction of a more rational design procedure, even if the resulting figures of probability of failure or of survival ("reliability") are considered as comparative rather than as absolute measures. When an acceptable failure rate or reliability figure is agreed upon for a certain category of structures, together with a well-defined procedure of reliability analysis, remaining aware of the fact that the computed figures are to be used for comparing different designs, different materials, different missions, etc., rather than to be taken at face value, such procedure represents a rational approach. The irrationality of the reliability statistician is in his belief that reliability figures of the order of 0.99, 0.999, or 0.9999 can be taken at their face value with respect to large structures.

The basis of the probabilistic approach to safety analysis is quite simple [1]. If it is assumed that the strength R of a statistical population of structures is given by a frequency distribution $f_R(R)$ and that the loads S applied to this population are defined by a load spectrum $f_s(S)$ the safety factor

$$\nu = \frac{R}{S} \tag{1}$$

is itself a statistical distribution function $f(\nu)$, being a quotient of two distribution functions. The probability of ultimate load failure which is the probability that $\nu < 1$

$$p_U = \int_{\nu=0}^{\nu=1} f(\nu)\, d\nu = F(\nu)_{\nu=1} \qquad (2)$$

depends on the relative location of the center regions (means, medians or modes) of R and S.

Metal structures subject to a large number of repeated loads of statistically variable intensity S may fail in either of two modes:

(a) by excessive deformation, instability or sudden fracture resulting from a single occurrence of an unexpectedly high, rare load intensity;

(b) by progressive damage produced by repeated loads of operational intensity, in the form of distributed micro-cracks coalescing into localized macro-cracks, terminated by the occurrence of a load of high, but not unexpected intensity by which the damaged structure is destroyed.

While the first mode is usually referred to as "ultimate load failure" and the second as "fatigue failure," the latter is, in essence, also an ultimate load failure involving the fatigue-damaged structure; it occurs, however, under a terminal load of considerably lower intensity and therefore much higher frequency of occurrence than the "ultimate load" producing failure in mode (a).

In this differentiation it is implied that the spectrum of operational loads which produce fatigue damage differs from the spectrum of "ultimate loads" which produces both the ultimate load and the fatigue failure in such a way that the latter cannot be obtained from the former by simple extrapolation towards very low probabilities of occurrence. However, the spectrum of ultimate loads can be considered as a spectrum of extreme values of large samples of operational loads. By this assumption a quantitative relation between the two load spectra could be established.

2. Reliability Function and Safety Factors

The distribution function $F_\nu(\nu)$ of the quotient $\nu = R/S$ is evaluated as the marginal density of the joint distribution of ν and S [2]:

$$F_\nu(\nu) = \int_0^\infty F_R(\nu S) f_S(S) dS \tag{3}$$

and therefore

$$p_U = F_\nu(1) = \int_0^\infty F_R(S) f_S(S) dS \tag{4}$$

For example, if R and S are both logarithmic normally distributed with the mean $\log \breve{R}$ and $\log \breve{S}$ and the standard deviation δ_R and δ_S, then

$$P_U = \Phi[-(\log \nu_o)/\delta] \tag{5}$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$, $\delta = (\delta_R^2 + \delta_S^2)^{1/2}$, $\breve{R}$ and $\breve{S}$ are the medians of R and S and $\nu_o = \breve{R}/\breve{S}$ is a measure of safety referred to as the "central safety factor" associated with logarithmic normal distributions.

The probability of failure p_U is not a direct measure of the safety of a structure subject to a random sequence of loads from the population $F_S(S)$ during its service period. Such a measure is its probability of survival, or the reliability function $L(N)$ under the N load applications constituting the sequence and is given by

$$L(N) = \int_0^{\infty} [F_S(R)]^N f_R(R) dR = \int_0^{\infty} [1 - \bar{F}_S(R)]^N f_R(R) dR \quad (6)$$

provided $F_R(R)$ and $F_S(S)$ are independent of N and $\bar{F}_S(R) = 1 - F_S(R)$.

Since the major contribution of the integral in Eq. (6) usually comes from those values of R for which $\bar{F}_S(R) << 1$, $[1 - \breve{F}_S(R)]^N \doteq 1 - N\bar{F}_S(R)$. Therefore,

$$\ell n L(N) \doteq \ell n [1 - Np_U] \doteq -Np_U \qquad \text{for } Np_U << 1.$$

Hence, for $Np_U << 1$, Eq. (6) can be approximated by

$$L(N) \doteq \exp(-Np_U) \quad (7)$$

Equation (7) is the well-known reliability function for chance failures where $(1/p_U) = T_U$ represents the "return number" of failures. $L(N)$ is the stochastic limit of the proportion of structures not failing under sequences of N randomly applied loads S each of which contributes to the gradually accumulating damage.

$p_U(N)$ can be expressed in terms of $L(N)$ in the form

$$p_U(N) = -d\,\ell n\, L(N)/d\,N \tag{8}$$

Eq. (8) relates the reliability function to the risk of failure, which can also be considered as the conditional probability that the structure which has survived $(N-1)$ load applications will fail at the N^{th} load application. In this interpretation, $p_U(N)$ is called the "risk function" or the "hazard function" and plays an important role in statistical analysis of fatigue failures. For chance failures according to Eq. (7) it is a constant $p_U(N) = 1/T_R$.

In conventional design it is usually assumed that the safety factor can be based on a maximum load and a minimum resistance. However, with the exception of non-statistical loads, such as the fluid or bulk-pressure in storage containers or floor-loads in warehouses, no absolute maximum can be specified; similarly, no absolute minimum of the resistance values can be usually known, except for the smallest observation in samples of finite size. Thus a "minimum resistance" R_{min} will always be associated with a finite probability p of not being attained and thus denoted by R_p while a "maximum load" S_{max} will be associated with a finite probability q of being exceeded and denoted by S_q, however small the probabilities p and q are selected. Thus $R_{min} = R_p = r_p R_o$ and $S_{max} = S_q = s_q S_o$ where R_o and S_o are central locations of the distributions of R and S (Fig. 1).

If R and S are logarithmic normally distributed with means $\log \breve{R}$ and $\log \breve{S}$ and standard deviations δ_R and δ_S where $\breve{R}$ and $\breve{S}$ are medians of R and S, the ratio r_p is related to p by

$$p = \Phi[(\log r_p)/\delta_R] \tag{9}$$

and similarly the ratio s_q is related to q by

$$q = \bar{\Phi}\,[(\log s_q)/\delta_S] \tag{10}$$

where $\bar{\Phi}(x) = 1 - \Phi(x)$.

The conventional safety factor $\bar{\nu}$ can now be defined by

$$\bar{\nu} = \frac{R_{min}}{S_{max}} = R_p/S_q = \nu_o r_p/s_q \tag{11}$$

and is thus related to the selected ratios r_p and s_q and to the central safety factor $\nu_o = \breve{R}/\breve{S}$. For a non-statistical maximum load $S_{max} = S^*$

$$\bar{\nu} = R_p/S^* = \nu_o r_p \qquad (12)$$

It is, in general, unlikely that the minimum value R_p can be specified with better approximation than $p = 0.1$: with samples consisting of 9 specimens the probability of values smaller than the smallest value of the sample is 0.1. Analysis of actual acceptance tests of structural steel has shown that about 10 per cent of the observed yield-stress values fall below the specified minimum.[3]

The specification of the maximum load S_q depends on the length of the time series of load observations from which it is to be determined. With a recurrence number of $(1/q)$ of loads exceeding S_q, the value of q can be selected as small as desired if the series of observations is long enough.

As an example, the ratios r_p and s_q have been computed from Eqs. (9) and (10) for $p = 0.1$ and $q = 0.1$, 0.01, 0.001 and 0.0001; p_f has also been evaluated as a function of ν_o with the aid of Eq. (5) for logarithmic normal distributions of R and S with the coefficient of variations in terms of the median $\sigma_R/\breve{R} = 0.5$ and $\sigma_S/\breve{S} = .20$ where σ_R and σ_S are the standard deviations of R and S. Then Eq. (11) is used to relate the conventional safety factor $\bar{\nu}$ to the probability of failure p_f.

The foregoing discussion is based on the assumption that the form of the distribution functions $f_R(R)$ and $f_S(S)$ is known not only near the center (where no significant distinction can usually be made between the different functions) but sufficiently far from the center to produce values of p_F of an acceptable order of magnitude. This is where the crux of the problem of reliability estimation for large structures lies: within this range the distribution functions, particularly of the strength, can never be known, since the number of necessary experiments is prohibitive. Hence the shape of the functions must be selected on the basis of physical-probabilistic reasoning and not purely statistical curve-fitting. Unless the functions are selected on this basis it is obvious that the resulting reliability figures have no physical meaning.

3. Reliability under Conditions of Fatigue

If it is assumed that the rate of damage per load cycle is proportional to the difference between the applied stress level and a "threshold stress" (endurance limit), the effective stress under constant load intensity S increases as the initial cross section A is reduced by progressive damage to $(A - A_r)$. Introducing $D = (1 - \frac{A_r}{A})$, the stress increases therefore as $(1 - D)^{-k}$, where $0 < D < 1$

is a measure of the fatigue damage and $1 < k < 2$ characterizes the effect of the reduced cross section on the resultant stress intensity. The damage rate can therefore be expressed by

$$\frac{dD}{dn} = f\left[\frac{S}{(1 - D)^k}\right] \qquad (13)$$

While practically all metal structures subject to repeated loads will show fatigue damage if the number of repetitions is large enough, fatigue is a significant design criterion only if the "return period" of fatigue failure under repeated variable load intensity is considerably shorter than the return period of ultimate load failure T_R. The safety factor of a structure subject to fatigue damage is no longer a stationary statistical variable but decreases with increasing number n of load repetitions which gradually reduce the resistance R to ultimate load failure. Hence, instead of Eq. (11) where ν_U is independent of n, ν is now a function of n

$$\nu_F = R(n)/S = \nu(n) \qquad (15)$$

through the fatigue damage $D(n)$ which expresses the reduction of the resistance R by changing the distribution $P_1(R)$ to a family of distributions

$$P_1\,[R(n)] = P_1\left[R[1 - D(n)]^k\right] \qquad (16)$$

where $D(0) < D(n) \leq D(N)$, with $D(0) = 0$ and $D(N) = 1$.

The distribution of

$$\frac{R}{S}\,[1 - D(n)]^k \;=\; \nu_U[1 - D(n)]^k \tag{17}$$

necessarily differs from the distribution of the quotient (R/S) because $[1 - D(n)]^k$ is not a constant but a statistical variable due to the statistical character of the damage function D(n). Only if the distributions of both ν_U and $[1 - D(n)]^k$ are logarithmic-normal, the distribution of ν_F is also logarithmic-normal.

Under the simplifying assumption of non-statistical linear damage accumulation D(n) the resulting relation between the distribution functions

$$P(\nu_U) \;=\; P\left[\frac{\nu_F}{(1 - D(n)]^k}\right] \tag{18}$$

implies that the probability of fatigue failure p_F at which $\nu_F \le 1$ is at the abscissa of the function $P(\nu_U)$ at which $\nu_U \le [1 - D(n)]^{-k}$. The distribution functions $P(\nu_U)$ computed under various assumptions for the distribution functions $P_1(R)$ and $P_2(S)$ and for the "central safety factor" of the design $\nu_U = \breve{R}/\breve{S}$, can be used to determine the probability of fatigue failure under the ultimate load spectrum S as a function $P_F[D(n)]$ of prior fatigue damage produced by the operational load spectrum. Hence ν_F has a distribution function $p[\nu(n)]$ the center of which gradually moves towards

$\nu \leq 1$, increasing the probability of failure p_F. Since $[1 - D(n)]^{-k} > 1$, the probabilities $p_F > p_U$.[4] The resulting risk function $r_u(N)$ is therefore not a constant but increases with N.

4. Conclusion

In conclusion I should like to say that I do not believe that a simple defense of conventional design methods is possible on the basis of the argument that what has been good in the past must be acceptable in the future. The probabilistic approach to safety analysis cannot be discredited on this basis. Let me read to you a statement appearing in our Civil Air Regulations, CAB, part 4b:

> (1) A limit load is a maximum load anticipated in <u>normal</u> conditions of operation.
>
> (2) An ultimate load is a limit load multiplied by an <u>appropriate</u> factor of safety.
>
> (3) The factor of safety is a design factor used to provide for the possibility of loads greater than those anticipated in normal conditions of operation and for uncertainties in design.

Do you believe that such a statement has a meaning that can be rationally interpreted?

I am not a reliability engineer but a structural designer who has attempted to use the extremely effective tools provided by the theory of probability to improve one part of the structural analysis that has been utterly neglected by the design engineer: safety analysis. It is time for the structural engineer to put his house in order, and this he must do himself unless he wants the reliability engineer to demolish it in order to erect on his place a structure made up of pure phantasy. A serious effort is needed to devise procedures of safety analysis which form an organic complement to load and stress-analysis, and which will permit an estimate of the reliability of any designed structure within rational limits. So far the community of structural engineers in all fields has not been in sympathy with this point of view. The aircraft industry is the first to be faced with a distorted image of the concept of reliability constructed by mathematical statisticians, and it will require a serious effort in research and design to reconstruct this image so that it becomes a useful tool of design rather than a form of sophisticated deceit.

List of Figures

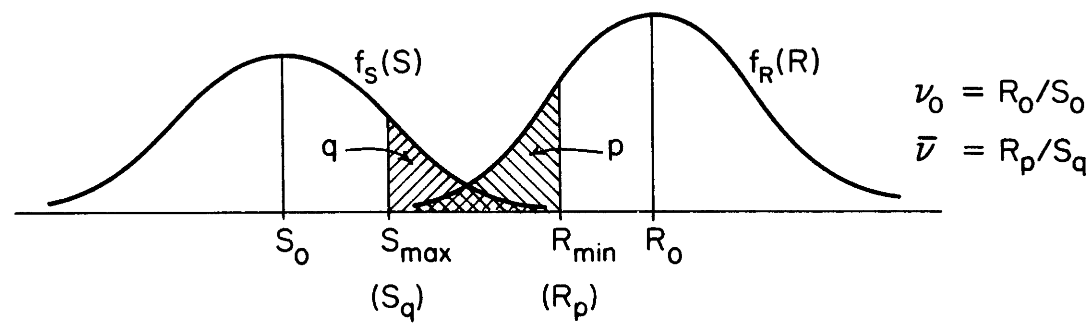

Fig. 1. Distributions of load $f_S(S)$ and of resistance $f_R(R)$ in relation to the safety factors ν_o and $\bar{\nu}$.