APPENDIX B

Derivation of the Lognormal related functions

For convenience of derivation, I assume the location parameter $t_0 = 0$.

The percentile of the 2-parameter Lognormal distribution t_p than can be directly determined from the CDF_L:

$$F_L(t) = \Phi\left[\ln\left(\frac{t - t_0}{\theta}\right)^{\rho}\right], \qquad \text{(B - 1)}$$

so,

$$z_p = \ln\left(\frac{t_p}{\theta}\right)^{\rho}$$

rearranging,

$$t_p = \theta \exp\left(\frac{z_p}{\rho}\right), \qquad \text{(B - 2)}$$

The median life is the 50^{th} percentile, hence substituting 0.5 into p yields:

$$Median = t_{0.5} = \theta \exp\left(\frac{z_{0.5}}{\rho}\right) = \theta \exp(0) = \theta \qquad \text{(B - 3)}$$

The mode is derived by setting $f_L'(t) = 0$, and solving for the maximum values of t.

$$f_L(t) = \frac{\rho}{\sqrt{2\pi}\ t}\exp\left(-\frac{\left[\ln\left(\frac{t}{\theta}\right)^{\rho}\right]^2}{2}\right),$$

$$\text{for} \quad \theta > 0,\ \rho > 0,\ t > 0\ . \qquad \text{(B - 4)}$$

$$f'(t) = \frac{-\rho}{t^2\sqrt{2\pi}}\exp\left\{-\frac{\left[\ln\left(\frac{t}{\theta}\right)^{\rho}\right]^2}{2}\right\} + \frac{\rho}{t\sqrt{2\pi}}\frac{d}{dt}\exp\left\{-\frac{\left[\ln\left(\frac{t}{\theta}\right)^{\rho}\right]^2}{2}\right\}$$

where

$$\frac{d}{dt}\exp\left\{\frac{-\left[\ln\left(\frac{t}{\theta}\right)^{\rho}\right]^2}{2}\right\} = \exp\left\{\frac{-\left[\ln\left(\frac{t}{\theta}\right)^{\rho}\right]^2}{2}\right\}\frac{d}{dt}\left\{\frac{-\left[\ln\left(\frac{t}{\theta}\right)^{\rho}\right]^2}{2}\right\}$$

$$= \exp\left\{\frac{-\left[\ln\left(\frac{t}{\theta}\right)^{\rho}\right]^2}{2}\right\}\left(\frac{-\rho^2}{2}\right)\frac{d}{dt}\left(\ln^2 t - 2\ln t\ln\theta + \ln^2\theta\right)$$

$$= \exp\left\{\frac{-\left[\ln\left(\frac{t}{\theta}\right)^{\rho}\right]^{2}}{2}\right\}\left(-\rho^{2}\right)\left(\frac{\ln t - \ln\theta}{t}\right)$$

For convenience of derivation, we set $c = \frac{\rho}{\sqrt{2\pi}}$ and $q = \frac{\left(\ln\left(\frac{t}{\theta}\right)^{\rho}\right)^{2}}{2}$

then

$$f'(t) = \frac{-1}{t^2} ce^{-q} + \frac{c}{t}\left[e^{-q}\left(-\rho^2\right)\left(\frac{\ln t - \ln\theta}{t}\right)\right]$$
$$= \frac{-c}{t^2} e^{-q}\left[1 + \rho^2(\ln t - \ln\theta)\right]$$
$$= -ce^{-q}\left[\frac{1+\rho^2(\ln t - \ln\theta)}{t^2}\right] \qquad \text{(B-5)}$$

Let the last term of equation (B-5) = 0 in order to solve for the maximum value i.e. the mode.

$$\left[\frac{1+\rho^2(\ln t - \ln\theta)}{t^2}\right] = 0,$$

so

$$\ln t = \ln\theta - \frac{1}{\rho^2}.$$

Finally the value of mode can be obtained.

$$t_{\text{mod}\,e} = \exp\left(\ln\theta - \frac{1}{\rho^2}\right) = \theta\exp\left[-\left(\frac{1}{\rho}\right)^2\right] = Mode \qquad \text{(B-6)}$$

For convenience sake, let $\omega = \exp\left(\frac{1}{\rho^2}\right)$

Hence,

$$Mode = \frac{\theta}{\omega} \qquad \text{(B-7)}$$

In order to determine the inflection point, the second derivative is needed.

$$f''(t) = -c\left[1+\rho^2(\ln t - \ln\theta)\right]\left(-2t^{-3}\right)e^{-q} + (-c)e^{-q}\left(\frac{\rho^2}{t^3}\right)$$

$$+\left(\frac{-c}{t^2}\right)\left[1+\rho^2(\ln t - \ln\theta)\right]\frac{d}{dt}e^{-q}$$

$$= \frac{ce^{-q}}{t^3}\left\{2\left[1+\rho^2(\ln t - \ln\theta)\right] - \rho^2 + \left[1+\rho^2(\ln t - \ln\theta)\right]\left[\rho^2(\ln t - \ln\theta)\right]\right\}$$

$$= \frac{ce^{-q}\rho^4}{t^3}\left(\frac{2}{\rho^4} + \frac{3\ln t}{\rho^2} - \frac{3\ln\theta}{\rho^2} - \frac{1}{\rho^2} + \ln^2 t - 2\ln t\ln\theta + \ln^2\theta\right) \qquad \text{(B-8)}$$

Equating the last term of equation (B-8) = 0, the inflection point can be determined.

$$\frac{2}{\rho^4} + \frac{3\ln t}{\rho^2} - \frac{3\ln\theta}{\rho^2} - \frac{1}{\rho^2} + \ln^2 t - 2\ln t\ln\theta + \ln^2\theta = 0$$

After rearranging,

$$\ln^2 t - 2\ln t \ln\theta + \frac{3\ln t}{\rho^2} + \left\{ \ln^2\theta - \frac{3\ln\theta}{\rho^2} + \left[\left(\frac{3}{2\rho^2}\right)\right]^2 \right\} = \frac{1}{\rho^2} + \frac{1}{4\rho^4}$$

$$\left[\ln t - \left(\ln\theta - \frac{3}{2\rho^2}\right)\right]^2 = \frac{1}{\rho^2} + \frac{1}{4\rho^4}$$

hence,

$$\ln t - \left(\ln\theta - \frac{3}{2\rho^2}\right) = \pm\sqrt{\frac{1}{\rho^2} + \frac{1}{4\rho^4}}$$

Finally the inflection points on the x axis can be obtained:

$$t = \exp\left[\left(\ln\theta - \frac{3}{2\rho^2}\right) \pm \frac{1}{\rho}\sqrt{1 + \frac{1}{4\rho^2}}\right] \qquad \text{(B - 9)}$$

The mean, variance, CV, SK, KU of the Lognormal distribution can be determined by moments method. The k^{th} moment about the origin is defined as:

$$E\left(x^k\right) = \int_0^\infty t^k f(t)dt$$

For the Lognormal distribution, the general expression of the k^{th} moment about the origin is given by [37].

$$E(x^k) = e^{\left(k \ln\theta + \frac{k^2}{2\rho^2}\right)} = \theta^k \sqrt{\omega^{k^2}}$$

The first moment can be easily seen as $E(x^k) = E(x)$ = Mean if k = 1, hence,

$$E(x) = Mean = \exp\left(\ln\theta + \frac{1}{2\rho^2}\right) = \theta\sqrt{\omega} \qquad \text{(B-10)}$$

.The variance is defined as:

$$VAR = E\left(x^2\right) - \left[E(x)\right]^2$$

This is the second moment minus the square of the first moment.

The second moment of the Lognormal distribution is:

$$E\left(x^2\right) = \theta^2 \omega^2$$

Therefore the variance of the Lognormal is obtained:

$$\begin{aligned} VAR &= \theta^2\omega^2 - \theta^2\omega \\ &= \theta^2\omega(\omega - 1) \end{aligned} \qquad \text{(B-11)}$$

The standard deviation (SD), therefore, is:

$$SD = \theta\sqrt{\omega(\omega - 1)} \qquad \text{(B-12)}$$

The coefficient of variation (CV) is defined as SD/Mean, thus,

$$CV = \frac{\theta\sqrt{\omega(\omega - 1)}}{\theta\sqrt{\omega}} = \sqrt{\omega - 1} \qquad \text{(B-13)}$$

The k^{th} moment about the mean is defined as:

$$E\left[x - E(x)\right]^k$$

For the Lognormal distribution, the general expression of the kth moment about the mean is given by [37].

$$E\left[x - E(x)\right]^k = \theta^k \sum_{i=0}^{k} (-1)^i \binom{k}{i} \exp\left\{\frac{\left[(k-i)^2 + i\right]}{2\rho^2}\right\}$$

$$= \theta^k \sum_{i=0}^{k} (-1)^i \frac{k!}{(k-i)!\,i!} \sqrt{\omega^{\left[(k-i)^2 + i\right]}}$$

The skewness is defined as the expected value of the third moment about the mean $E\left[x - E(x)\right]^3$.

The coefficient of skewness (CS) is defined as $\frac{E\left[x - E(x)\right]^3}{VAR^{\frac{3}{2}}}$:

$$CS = \frac{\theta^3\left(\frac{3!}{3!0!}\sqrt{\omega^9} - \frac{3!}{2!1!}\sqrt{\omega^5} + \frac{3!}{1!2!}\sqrt{\omega^3} - \frac{3!}{0!3!}\sqrt{\omega^3}\right)}{\left[\theta^2\omega(\omega-1)\right]^{\frac{3}{2}}}$$

$$= \frac{\sqrt{\omega^9} - 3\sqrt{\omega^5} + 2\sqrt{\omega^3}}{\left[\omega(\omega-1)\right]^{\frac{3}{2}}}$$

$$= \frac{\omega^{\frac{3}{2}}\left(\omega^3 - 3\omega + 2\right)}{\left[\omega(\omega-1)\right]^{\frac{3}{2}}}$$

$$= \frac{\left(\omega^2 - 2\omega + 1\right)(\omega+2)}{(\omega-1)^{\frac{3}{2}}}$$

$$= \sqrt{\omega-1}(\omega+2) \qquad \text{(B-14)}$$

The kurtosis is defined as the expected value of the fourth moment about the mean $E[x - E(x)]^4$.

The coefficient of kurtosis(CK) is defined as $\frac{E[x-E(x)]^4}{VAR^{\frac{4}{2}}}$:

$$CK = \frac{\theta^4\left(\frac{4!}{4!0!}\sqrt{\omega^{16}} - \frac{4!}{3!1!}\sqrt{\omega^{10}} + \frac{4!}{2!2!}\sqrt{\omega^6} - \frac{4!}{1!3!}\sqrt{\omega^4} + \frac{4!}{0!4!}\sqrt{\omega^4}\right)}{\left[\theta^2\omega(\omega-1)\right]^2}$$

$$= \frac{\sqrt{\omega^{16}} - 4\sqrt{\omega^{10}} + 6\sqrt{\omega^6} - 3\sqrt{\omega^4}}{\left[\omega(\omega-1)\right]^2}$$

$$= \frac{\omega^2\left(\omega^6 - 4\omega^3 + 6\omega - 3\right)}{\left[\omega(\omega-1)\right]^2}$$

$$= \frac{(\omega-1)^2\left(\omega^4 + 2\omega^3 + 3\omega^2 - 3\right)}{(\omega-1)^2}$$

$$= \omega^4 + 2\omega^3 + 3\omega^2 - 3 \qquad \text{(B-15)}$$