APPENDIX A

Derivation of the Weibull related functions

For convenience of derivation, I assume the location parameter $t_0 = 0$.

The percentile of the 2-parameter Weibull distribution t_p than can be directly determined from the CDF_W:

$$F_W(t) = 1 - e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)}. \tag{A-1}$$

Rearranging equation (A-1), we obtain:

$$\left(t_p\right)_W = \eta\left(\ln\frac{1}{1-p}\right)^{\frac{1}{\beta}}. \tag{A-2}$$

The median is the 50th percentile, hence substituting 0.5 into p yields:

$$Median_W = \eta(\ln 2)^{\frac{1}{\beta}}. \tag{A-3}$$

The mode is derived by setting $f'_W(t) = 0$, and solving for the maximum values of t.

$$f_W(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)}, \tag{A-4}$$

$$f_W'(t) = \frac{\beta}{\eta}(\beta - 1)\left[\left(\frac{t}{\eta}\right)^{\beta - 2}\right]\left(\frac{d\left(\frac{t}{\eta}\right)}{d\ t}\right)e^{\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)}$$

$$+\frac{\beta}{\eta}\left[\left(\frac{t}{\eta}\right)^{\beta-1}\right]e^{-\left(\frac{t}{\eta}\right)^{\beta}}\left[\frac{d\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)}{d\ t}\right]$$

$$= \frac{\beta}{\eta^2}e^{-\left(\frac{t}{\eta}\right)^{\beta}}\left[(\beta-1)\left(\frac{t}{\eta}\right)^{\beta-2} - \left(\frac{t}{\eta}\right)^{\beta-1}\beta\left(\frac{t}{\eta}\right)^{\beta-1}\right]$$

$$= \frac{\beta}{\eta^2}e^{-\left(\frac{t}{\eta}\right)^{\beta}}\left(\frac{t}{\eta}\right)^{\beta-2}\left[(\beta-1) - \left(\frac{t}{\eta}\right)^{\beta}\beta\right]$$

For convenience of derivation, we set $q = \frac{t}{\eta}$ then

$$f'(t) = \frac{\beta}{\eta^2}e^{-q^{\beta}}q^{\beta-2}\left(-\beta q^{\beta} + \beta - 1\right) \qquad \text{(A-5)}$$

Let the last term of equation (A-5) = 0 to solve for the maximum value i.e. the mode.

$$-\beta q^{\beta} + \beta - 1 = 0$$

$$\frac{\beta - 1}{\beta} = \left(\frac{t}{\eta}\right)^{\beta}$$

Finally the value of the mode can be obtained if β ≥ 1

$$t_{mode} = \eta\left(\frac{\beta-1}{\beta}\right)^{\frac{1}{\beta}} = \text{Mode} \qquad \text{(A-6)}$$

In order to determine the inflection point, the second derivative is needed.

$$f''(t) = \frac{\beta}{\eta^3} e^{-q^\beta} q^{\beta-3}\left[\left(\beta q^\beta\right)^2 - (3\beta-3)\beta q^\beta + (\beta-2)(\beta-1)\right] \qquad \text{(A-7)}$$

Equating the last term of equation (A-7) = 0, the inflection point can be obtained by solving for the root of the equation.

$$\left(\beta q^\beta\right)^2 - (3\beta-3)\beta q^\beta + (\beta-2)(\beta-1) = 0$$

$$q^\beta = \frac{3(\beta-1) \pm \sqrt{(5\beta-1)(\beta-1)}}{2\beta}$$

Finally the inflection points on the x axis can be obtained from:

$$t = \sqrt[\beta]{\frac{3(\beta-1) \pm \sqrt{(5\beta-1)(\beta-1)}}{2\beta}}\ \eta \qquad \text{(A-8)}$$

It is clear that the inflection points exist only when β > 1. When β = 2, one of the inflection points is always 0.

The k^{th} moment about the origin is defined as:

$$E\left(x^{k}\right) = \int_{0}^{\infty} t^{k} f(t)dt$$

For the Weibull distribution, the general expression of the k[th] moment about the origin is given by [37].

$$E\left(x^{k}\right) = \eta^{k}\Gamma\left(1+\frac{k}{\beta}\right)$$

Let $\gamma_k = \Gamma\left(1+\frac{k}{\beta}\right), k = 0,\ 1,\ 2,\ 3,\ 4\ldots$

then $E\left(x^{k}\right) = \eta^{k}\gamma_k$

The first moment can be easily seen as $E(x^k) = E(x) = Mean$ if $k = 1$, hence,

$$E(x) = Mean = \eta\gamma_1 = \eta\Gamma\left(1+\frac{1}{\beta}\right)\ . \qquad \text{(A - 9)}$$

The variance of the Weibull distribution is defined as:

$$VAR = E\left(x^{2}\right) - \left[E(x)\right]^{2}$$

This is the second moment minus the square of the first moment.

The second moment is:

$$E\left(x^{2}\right) = \eta^{2}\gamma_2 = \eta^{2}\Gamma\left(1+\frac{2}{\beta}\right)$$

Therefore the variance is obtained:

$$VAR = \eta^2\gamma_2 - (\eta\gamma_1)^2 = \eta^2(\gamma_2 - \gamma_1{}^2)$$
$$= \eta^2\left\{\Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\} \qquad \text{(A-10)}$$

The standard deviation (SD), therefore, is:

$$SD = \eta\sqrt{\gamma_2 - \gamma_1{}^2} \qquad \text{(A-11)}$$

The coefficient of variation (CV) is defined as SD/Mean, thus,

$$CV = \frac{\eta\sqrt{\gamma_2 - \gamma_1{}^2}}{\eta\gamma_1} = \sqrt{\frac{\gamma_2}{\gamma_1{}^2} - 1} \qquad \text{(A-12)}$$

The k^{th} moment about the mean is defined as:

$$E[x - E(x)]^k$$

For the Weibull distribution, the general expression of the k^{th} moment about the mean is given by [37].

$$E[x - E(x)]^k = \eta^k \sum_{i=0}^{k} (-1)^i \binom{k}{i} \Gamma\left(1+\frac{k-i}{\beta}\right)\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^i$$
$$= \eta^k \sum_{i=0}^{k} (-1)^i \binom{k}{i} \gamma_{k-i}\gamma_1{}^i$$
$$= \eta^k \sum_{i=0}^{k} (-1)^i \frac{k!}{(k-i)!\,i!} \gamma_{k-i}\gamma_1{}^i$$

The skewness is defined as the expected value of the third moment about the mean $E[x - E(x)]^3$.

The coefficient of skewness (CS) is defined as $\dfrac{E\left[x-E(x)\right]^3}{VAR^{\frac{3}{2}}}$:

$$CS=\frac{\eta^3\left(\frac{3!}{3!0!}\gamma_3\gamma_1{}^0-\frac{3!}{2!1!}\gamma_2\gamma_1{}^1+\frac{3!}{1!2!}\gamma_1\gamma_1{}^2-\frac{3!}{0!3!}\gamma_0\gamma_1{}^3\right)}{\left[\eta^2\left(\gamma_2-\gamma_1{}^2\right)\right]^{\frac{3}{2}}}$$

$$=\frac{\gamma_3-3\gamma_2\gamma_1+2\gamma_1{}^3}{\left[\left(\gamma_2-\gamma_1{}^2\right)\right]^{\frac{3}{2}}} \qquad \text{(A-13)}$$

The kurtosis is defined as the expected value of the fourth moment about the mean $E\left[x-E(x)\right]^4$.

The coefficient of kurtosis(CK) is defined as $\dfrac{E\left[x-E(x)\right]^4}{VAR^{\frac{4}{2}}}$:

$$CK=\frac{\eta^4\left(\frac{4!}{4!0!}\gamma_4\gamma_1{}^0-\frac{4!}{3!1!}\gamma_3\gamma_1{}^1+\frac{4!}{2!2!}\gamma_2\gamma_1{}^2-\frac{4!}{1!3!}\gamma_1\gamma_1{}^3+\frac{4!}{0!4!}\gamma_0\gamma_1{}^4\right)}{\left[\eta^2\left(\gamma_2-\gamma_1{}^2\right)\right]^2}$$

$$=\frac{\gamma_4-4\gamma_3\gamma_1+6\gamma_2\gamma_1{}^2-3\gamma_1{}^4}{\left(\gamma_2-\gamma_1{}^2\right)^2} \qquad \text{(A-14)}$$