AFML-TR-77-170

THE CONCEPT OF MAXIMUM RELIABILITY SELECTION OF UNKNOWN DISTRIBUTION PARAMETERS.

Avenue d'Albigny, 9 bis
74000 Annecy
France

September 1977

TECHNICAL REPORT AFML-TR-77-170
Final Report for Period 1 October 1973 - 15 December 1973

AIR FORCE MATERIALS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433

NOTICE

This technical report has been reviewed and is approved for publication.

ROBERT C. DONAT
Metals Behavior Branch
Metals and Ceramics Division

FOR THE COMMANDER

GAIL E. EICHELMAN
Acting Chief
Metals Behavior Branch
Metals and Ceramics Division

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFML-TR-77-170	2. GOVT ACCESSION NO.
	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): The Concept of Maximum Reliability Selection of Unknown Distribution	5. TYPE OF REPORT & PERIOD COVERED: Final report, for period 1 Oct 1970 - 15 Dec 1973
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Waloddi Weibull	8. CONTRACT OR GRANT NUMBER(s): F44620-73-C-0066
9. PERFORMING ORGANIZATION NAME AND ADDRESS: AFML/LLN, Wright-Patterson AFB, OH 45433	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 7351-06, 62102F 687351
11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE: September 1977
	13. NUMBER OF PAGES: 17
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

distribution parameter selection
structural reliability
distribution functions

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The classical method of estimating an unknown parameter of a given distribution function by means of a unique function of the observations has been replaced by a procedure which consists in deciding between several possible values of the parameter by use of a test statistic, called the selector. Its merit is appraised by use of a new concept, the reliability of the selector, which is equal to the probability of selecting the true value of the parameter. It has been proved that the maximum likelihood estimation method possesses maximum reliability.

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Pertinent formulas have been developed and applied to the Weibull distribution.

FOREWORD

The research work reported herein was conducted by Prof. Dr. Waloddi Weibull, Avenue d'Albigny, 9 bis, 74000 Annecy, France under USAF Contract No. F44620-73-C-0066. This contract, which was initiated under Project No. 7351, "Metallic Materials", Task 735106, "Behavior of Metals", was administered by the European Office of Aerospace Research. The work was monitored by the Metals and Ceramics Division, Air Force Materials Laboratory, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, under the direction of Mr. W. J. Trapp, AFML/LL.

This report covers work conducted during the period 1 October 1973 to 15 December 1973. The manuscript was submitted by the author for publication in January 1974.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

1. INTRODUCTION

Consider a distribution of the continuous type with the density function $f(x;\alpha)$, where α is a parameter. The values $x_1,\ldots,x_N$ obtained in N independent drawings from the distribution, which will be called the observations, are independent random variables, all of which have the same density function $f(x;\alpha)$. Each particular sample X_N will be represented by a definite point $X_N=(x_1,\ldots,x_N)$ in the sample space R_N of the variables $x_1,\ldots,x_N$. The probability element of the joint distribution of the observations is

$$L(x_1,\ldots,x_N)\,dx_1\ldots dx_N = f(x_1;\alpha)\ldots f(x_N;\alpha)\,dx_1\ldots dx_N \tag{1}$$

which is equal to the probability that the sample point X_N falls within the N-dimensional interval $dx_1\ldots dx_N$.

The function L is known as the likelihood function of the sample X_N.

The classical method of estimating the unknown parameter α by means of the observations consists in using a unique function $\hat{\alpha}=\hat{\alpha}(x_1,\ldots,x_N)$ of the observations as an estimate of α. The merit of this estimator is appraised by its variance. Under certain general conditions, the smallest possible value of this variance is given by

$$D^2_{min}(\hat{\alpha}) = 1/N\int_{-\infty}^{\infty}(\partial \ln f(x;\alpha)/\partial\alpha)^2 \cdot f(x;\alpha)\,dx \tag{2}$$

The ratio between this minimum value and the actual variance of $\hat{\alpha}$ is called the efficiency of $\hat{\alpha}$.

The procedure of estimating an unknown parameter α of a given distribution function by means of the observations will now be considered from a somewhat different aspect, viz. as a process of deciding between several possible values of α.

Any procedure of selecting one of a set of competing hypotheses consists in choosing a unique function of the observations, which will be called the selector, and a set of acceptance regions, one for each of the hypotheses. The merit of the selector will be appraised by means of a new concept, called the reliability of the selector, as will be demonstrated in the following.

2. THE RELIABILITY OF A SELECTOR

2.1 A Finite Number of Hypotheses

Any selector $T(t_1,\ldots,t_k)$ is a unique function of k test values. Each particular value of T will be represented by a definite point in the k-dimensional space A of the test values $t_1,\ldots,t_k$. If $k=1$ the selector is said to be univariate, if $k=2$, bivariate, etc.

Consider the case that we have to select one of j hypotheses $H1,\ldots,Hj$ by means of the selector T. If Hi is the true hypothesis, then T has a particular density function, which will be denoted by $f_i(t_1,\ldots,t_k)$. We now have to choose j acceptance regions $A1,\ldots,Aj$, which are parts of the space A without common points. We have

$$A - \Sigma A_i \geq 0 \qquad (3)$$

If the sign of inequality holds, then the non-empty region $(A - \Sigma A_i)$ will be the acceptance region of the hypothesis that none of the j hypotheses is true.

The selection rule now becomes, that, if the particular value of T, the test point, falls within Ai, then the hypothesis Hi is accepted and all the other hypotheses rejected.

Let us now, for a moment, suppose that Hi is the true hypothesis, then we will state this fact, that is, we are making a correct selection, each time we obtain a test point $(t_1,\ldots,t_k)$ which falls within the region Ai. The probability of this event, denoted by PHi, is given by

$$PHi = \int_{Ai} f_i(t_1,\ldots,t_k)dt_1\ldots dt_k \qquad (4)$$

It is obvious that this probability depends on the choice of Ai. If we, for instance, put $A_i = A$, then $PHi = 1$, but then all other regions and probabilities will be equal to zero.

The proper choice of the acceptance regions follows certain rules, which are indicated below.

Let us now suppose that we can give preference to none of the competing hypotheses. If we then repeat the selection procedure many times, then it is reasonable to assume that each hypothesis will have the probability $1/j$ of occurring, and the probability of selecting the true hypothesis will be given by the arithmetic mean of all probabilities PHi, that is,

$$PS = \Sigma PHi/j \tag{5}$$

Also PS depends on the choice of the acceptance regions, and it is required to define the particular set Ai, which maximizes PS. This problem will now be examined for several different alternatives.

2.1.1 Univariate Selectors

These selectors have one-dimensional density functions $f_i(t)$. For the simple case of two hypotheses only $(j = 2)$ let $f_1(t)$ and $f_2(t)$ be represented by the graphs in Fig.1.

If we now arbitrarily choose as the critical point t_c, which separates the regions $A1 = (-\infty, t_c)$ and $A2 = (t_c, \infty)$, then from (4)

$$PH1 = \int_{-\infty}^{t_c} f_1(t)\,dt \qquad PH2 = \int_{t_c}^{\infty} f_2(t)\,dt \qquad PS = (PH1 + PH2)/2 \tag{6}$$

These formulas are valid for any choice of acceptance regions, but a moment's reflection will show that PS will be maximized, if, but only if, we take as the critical point c_{12}, which is the abscissa of the intersection between the two density functions, and thus defined by

$$f_1(c_{12}) = f_2(c_{12}) \tag{7}$$

It can be concluded that these regions may also be defined as follows:

$$\begin{aligned}&\text{A1 contains all points of the space of } T \text{ such that } f_1(t) > f_2(t)\\&\text{A2 contains all points of the space of } T \text{ such that } f_2(t) > f_1(t)\end{aligned} \tag{8}$$

This maximum value of PS will be denoted by RS and called the reliability of the selection.

Equs. (6) and (8) are valid also in the more complicated cases, when there are more than one intersection between the density functions, as illustrated in Fig.2, where A2 is composed of the two intervals A2a and A2b.

The extension to any finite number of hypotheses is immediate.

The reliability RS can be put in relation to the concept of decision power DP, introduced in Sci.Rep.Nr.3 of Contract F61052-69-C-0029 [1] and defined by

$$DP = 1 - \text{Prob}(E1) - \text{Prob}(E2) = \int_t (f_1(t) - f_2(t))\,dt \tag{9}$$

where Prob(E1) = the probability of rejecting the true hypothesis and Prob(E2) = the probability of accepting a false hypothesis.

With the notations in Fig.3 we have for $j = 3$

$$PH1 = 1 - c_1 \quad PH2 = 1 - b_1 - c_2 \quad PH3 = 1 - b_2$$

$$DP(1,2) = 1 - b_1 - c_1 \quad DP(2,3) = 1 - b_2 - c_2$$

After some obvious calculations we arrive at

$$RS = (1 + DP(1,2) + DP(2,3))/3 \tag{10}$$

The extension to any number of hypotheses is immediate. If the arithmetic mean of the $(j-1)$ DP-values is denoted by E(DP), then

$$RS = (1 + (j-1)E(DP))/j \tag{11}$$

from which it can be concluded that with increasing j

$$RS \longrightarrow E(DP) \tag{12}$$

2.1.2 Multivariate Selectors

In accordance with the preceding argumentation it follows that for multivariate selectors we have

$$PHi = \int_{A_i} f_i(t_1,\ldots,t_k)\,dt_1 \ldots dt_k \quad \text{and} \quad RS = \Sigma PHi/j \tag{13}$$

where Ai contains all points of the k-dimensional space of T satisfying the inequality

$$f_i(t_1,\ldots,t_k) > f_h(t_1,\ldots,t_k) \quad (h \neq i) \tag{14}$$

2.2 An Infinite Number of Hypotheses

The preceding formulas will now be extended to the case of an infinite number of hypotheses, a problem which arises, when we have to select the true value of an unknown parameter α, which can take any value belonging to a non-degenerate interval.

In this particular case, the coordinates $t_1,\ldots,t_k$ of the selector are unique functions $t_i = g_i(x_1,\ldots,x_N)$ of the observations x_i.

The study will be started with the most simple selector $T = X_N$, that is, taking the sample point as the test point without any transformations, which implies $t_i = x_i$ and $k = N$.

Let Hi be the hypothesis that α_i is the true value of the unknown parameter α. The density function f_i will then be given by

$$f_i(x_1,\ldots,x_N) = f(x_1;\alpha_i)\ldots f(x_N;\alpha_i) \tag{15}$$

2.2.1 Univariate Selectors

If only one observation is available, then the density functions becomes $f(x;\alpha_i)$. Now let the density functions $f(x;\alpha-d\alpha)$, $f(x;\alpha)$ and $f(x;\alpha+d\alpha)$ be represented by the graphs in Fig.4. Maximum reliability is attained only if we choose the acceptance region $A\alpha=(c_1,c_2)$, defined by

$$f(c_1;\alpha-d\alpha)=f(c_1;\alpha) \quad \text{and} \quad f(c_2;\alpha)=f(c_2;\alpha+d\alpha) \tag{16}$$

For small $d\alpha$ we may put

$$f(c_1;\alpha-d\alpha) = f(c_1;\alpha)-f'_\alpha(c_1;\alpha)\,d\alpha$$

$$f(c_2;\alpha+d\alpha) = f(c_2;\alpha)+f'_\alpha(c_2;\alpha)\,d\alpha$$

where

$$f'_\alpha(x;\alpha) = \partial f(x;\alpha)/\partial\alpha \tag{17}$$

from which it follows that

$$f'_\alpha(c_1;\alpha) = f'_\alpha(c_2;\alpha) = f'_\alpha(c;\alpha) = 0 \tag{18}$$

that is, when $d\alpha \rightarrow 0$, then c_1 and c_2 tend to the same value c, which is the abscissa of a point common to $f(x;\alpha)$ and the envelop of the family of the density functions, as indicated in Fig.4. This result implies that the acceptance region (c_1,c_2) degenerates into the point c and the selection rule becomes that, if we have a single observation x_1, then we will select as the true value of α the particular value $\hat{\alpha}$, which is given by

$$\partial f(x_1;\hat{\alpha})/\partial\alpha = 0 \tag{19}$$

Observing that $f(x;\alpha)$ is the likelihood function of a sample X_N of size $N=1$, it follows that $\hat{\alpha}$ is identical with the maximum likelihood estimate, which thus has been proved to have maximum reliability in this particular case.

The selection rule (19) may also be put in the form

$$\partial \ln f(x_1;\hat{\alpha})/\partial\alpha = 0 \tag{20}$$

Let us now suppose that α can take a very large number of discrete, equidistant (distance $= d\alpha$) values α_i. As demonstrated in earlier publications, DP may then be replaced by the estimation power EP. The reliability of X_1 will then be given by

$$RS = E(EP(\alpha))\, d\alpha \tag{21}$$

From (9) it may be derived that

$$EP(\alpha) = \int_{+} f'_{\alpha}(x;\alpha)\, dx \tag{22}$$

where the integration includes all points x with positive values of f'_{α}, as indicated by the $+$ sign.

Since $EP(\alpha)$ is a function of α, typical for each value of α, the mean $E(EP)$ may be replaced by an integral. It will, however, be preferable, as being more informative, to use the $EP(\alpha)$-function itself as a measure of the reliability of a selector, as will be illustrated in the sequel.

The question now arises whether it will be possible to increase the reliability by introducing a transformation

$$y = g(x) \tag{23}$$

of the observations x.

Two necessary conditions will be imposed upon the function $g(x)$:

1) there must be a uniquely defined y correlated with each x.

2) no two of the transformed acceptance regions may have common points.

These two conditions are satisfied, if the function $g(x)$ defines a biunique mapping of the domain of y onto that of x, which holds if $g(x)$ is monotone, i.e. steadily increasing or steadily decreasing as x increases, as illustrated in Fig.4. If α is the true value, then the probability of selecting it is equal to the area of the shaded region, that is, to the probability that a value x drawn from a distribution with the density function $f(x;\alpha)$ falls within the interval (c_1,c_2). With each such value there is always correlated a value y which falls within the transformed acceptance region $(g(c_1),g(c_2))$, so it can be concluded that RS is invariant under any acceptable transformation. Consequently no improvement of the reliability is possible by means of transformations of the observations.

2.2.2 Bivariate and Multivariate Selectors

Let us now suppose that two observations are available. Taking X_2 as the selector, the density function corresponding to the hypothesis $H\alpha_i$ that α_i is the true value of α, will be given by

$$f_i(x_1,x_2;\alpha) = f(x_1;\alpha_i) \cdot f(x_2;\alpha_i) \tag{24}$$

Comparing three adjacent density functions, as in the preceding, corresponding to $\alpha - d\alpha$, α and $\alpha + d\alpha$, it follows that, when $d\alpha \longrightarrow 0$, the acceptance region $A\alpha$ degenerates into a curve in the x_1,x_2-plane.

The selection rule then becomes that, if we have two observations x_1 and x_2, then we will select as the true value of α the particular value $\hat{\alpha}$, which satisfies the condition

$$\partial[\ln f(x_1;\hat{\alpha}) + \ln f(x_2;\hat{\alpha})]/\partial\alpha = 0 \tag{25}$$

The extension to any number of observations is immediate, being

$$\partial[\ln f(x_1;\hat{\alpha}) + \ldots + f(x_N;\hat{\alpha})]/\partial\alpha = 0 \tag{26}$$

The selected value $\hat{\alpha}$ is identical with the maximum likelihood estimate.

The estimation power $EP(\hat{\alpha})$ will be given by

$$EP(\hat{\alpha}) = \int_{+}[\partial(f(x_1;\hat{\alpha})\ldots f(x_N;\hat{\alpha})/\partial\alpha]dx_1\ldots dx_N \tag{27}$$

where the integration is taken over all points with a positive value of the partial derivative, as indicated by the + sign.

It can be proved that $EP(\hat{\alpha})$ is invariant under any acceptable transformation $y_i = g(x_i)$, of the observations, which implies that no improvement of the reliability can be made in this way.

Let us now examine the effect of rearranging the elements $x_1 \ldots x_N$ of the sample in ascending order of magnitude, denoting them by $x_{(1)}, \ldots, x_{(N)}$ and calling them the order statistics in the sample.

The probability element of the joint distribution of an arbitrary set of order statistics is given by Sarhan & Greenberg [2]. In particular we have, if all order statistics are taken, the density function

$$N! [f(x_{(1)}) \ldots f(x_{(N)})] \tag{28}$$

which differs from the unarranged sample only by the factor $N!$. The selection rule, indicated by equ.(26), will thus result in the same selected value $\hat{\alpha}$.

The introduction of the order statistics has, however, the advantage of making it possible to censor or truncate the sample. We may even use a single order statistic of the sample. The estimation power $EP(\hat{\alpha})$ depends very much on the order number, thus indicating where the information is located within the sample.

The preceding general formulas will now be applied to the Weibull distribution and further developed.

3. APPLICATION TO THE WEIBULL DISTRIBUTION

3.1 One Unknown Parameter

A single unknown parameter may be determined by means of a single observation, as will now be demonstrated. The density function of the selector X_1 is given by

$$f(x, m, \beta, \mu) = (m/\beta)\, z^{m-1} e^{-z^m} \tag{29}$$

where

$$z = (x - \mu)/\beta \tag{30}$$

and

$m = 1/\alpha$ = the shape parameter

β, μ = the scale and the location parameter

From (29) we have

$$\ln f(x) = \ln m - \ln \beta + (m-1) \ln z - z^m \tag{31}$$

3.1.1 The Parameter $\alpha = 1/m$ Unknown, β and μ Known

From (31) it follows that

$$\partial \ln f(x)/\partial \alpha = -m^2 \, \partial \ln f(x)/\partial m = -m(1 + \ln z^m - z^m \ln z^m) \tag{32}$$

Introducing

$$\left.\begin{array}{c} u = z^m \qquad z = u^\alpha \\ dx = \alpha.\beta.u^{\alpha-1} du \\ \alpha.\beta.f(x) = u^{1-\alpha}.e^{-u} \;;\; f(x)\,dx = e^{-u}\,du \end{array}\right\} \tag{33}$$

we have

$$-\alpha \, \partial \ln f(x)/\partial \alpha = t(u) = 1 + \ln u - u.\ln u \tag{34}$$

Some values of the function $t(u)$ are listed in Table 1. We have $t(u_a) = t(u_b) = 0$ for

$$u_a = 0.25924 \;;\quad u_b = 2.23893$$

Hence, if a single observation x_1 is available, then it follows from equ.(20) that the selected value $\hat{m}$ is given by

$$x_1^{\hat{m}_1} = 0.25924 \quad \text{or} \quad x_1^{\hat{m}_2} = 2.23893$$

or

$$1/\hat{m}_1 = \hat{\alpha}_1 = -\ 1.70562 \log x_1 \quad \text{or} \quad 1/\hat{m}_2 = \hat{\alpha}_2 = 2.85683 \log x_1 \tag{35}$$

Since always $\alpha > 0$, the value $\hat{\alpha}_1$ is used, when $x_1 < 1$, and $\hat{\alpha}_2$, when $x_1 > 1$.

The estimation power of $\hat{\alpha}$ is from equ.(22) given by

$$EP(\hat{\alpha}) = \int_{+} \partial f(x)/\partial\alpha)\, dx = \int_{+} (\partial \ln f(x)/\partial \alpha) f(x)\, dx$$

Thus

$$\alpha . EP(\hat{\alpha}) = \int_{u_a}^{u_b} (1 + \ln u - u . \ln u) e^{-u}\, du \tag{36}$$

Observing that

$$d(u.\ln u . e^{-u}) = (1 + \ln u - u.\ln u) e^{-u} du$$

it follows that

$$\alpha . EP(\hat{\alpha}) = \Big/_{u_a}^{u_b} (u.\ln u . e^{u} = 0.46237 \tag{37}$$

<u>3.1.2 The Parameter β Unknown, α and μ Known</u>

From (31) we have after some easy calculations

$$\alpha . \beta\ \partial \ln(f(x)/\partial\ \beta = u - 1 \tag{38}$$

The selection rule then becomes

$$u = ((x_1 - \mu)/\hat{\beta})^m = 1$$

or

$$\hat{\beta} = x_1 - \mu \tag{39}$$

The estimation power $EP(\hat{\beta})$ is given by

$$\alpha . \beta . EP(\beta) = \int_{1}^{\infty} (u - 1)\, e^{-u}\, du = 0.36788$$

or

$$EP(\hat{\beta}) = 0.36788/\alpha . \beta \tag{40}$$

3.1.3 The Parameter μ Unknown, α and β Known

From (31) we have after some easy calculations

$$\alpha.\beta\, \partial \ln f(x)/\partial u = (u-(1-\alpha))/u^{\alpha} \tag{41}$$

The selected value $\hat{\mu}$ will be given by

$$u = ((x_1-\hat{\mu})/\beta)^m = 1-\alpha$$

or

$$\hat{\mu} = x_1 - \beta(1-\alpha)^{\alpha} \tag{42}$$

The estimation power $EP(\hat{\mu})$ is given by

$$\alpha.\beta.EP(u) = \int_0^{1-\alpha} (1-\alpha-u)\,u^{\alpha}.\,e^{-u}\,du \tag{43}$$

$$= 0 \quad \text{for} \quad \alpha = 1$$

$$= 0.36788 \text{ for } \alpha = 0$$

3.2 All Parameters Unknown

Maximum reliability is attained, if we choose X_N as the selector. Its density function L is given by equ (1). Introducing equ.(29) we have

$$L = \prod_{i=1}^{N} m(x_i - \mu)^{m-1}.\,e^{-(x_i-\mu)^m/\beta^m}/\beta^m \tag{44}$$

The selected values $\hat{m}$, $\hat{\mu}$ and $\hat{\beta}$ are obtained by equating to zero the partial derivatives, that is, by solving the system of equations

$$\partial L/\partial m = 0; \partial L/\partial \mu = 0 \quad \text{and} \quad \partial L/\partial \beta = 0 \tag{45}$$

From the last equation the value $\hat{\beta}$ will be given by

$$\hat{\beta}^m = \Sigma(x_i - \mu)^m/N \tag{46}$$

Introducing (46) into (44) and neglecting factors depending on N only, we arrive at

$$L = \prod_{i=1}^{N} m(x_i - \mu)^{m-1} / \Sigma(x_i - \mu)^m \tag{47}$$

In the particular case when $\mu = 0$ we have

$$L = \prod_{i=1}^{N} m \cdot x_i^{m-1} / \Sigma x_i^m \tag{48}$$

Instead of solving the system of equs.(45), it has been found convenient to compute L in equ.(47) for a properly chosen set of m and μ-values and to select the particular pair $\hat{m}$, $\hat{\mu}$ which maximizes L.

To this purpose the computer program 6/73 has been written and applied to a large number of samples of fatigue test data collected at the Boeing Company, as will be reported elsewhere.

The computing time for a complete evaluation of such samples of size $N = 10$ is only about one second.

4. REFERENCES

1. Weibull, W.: "Outline of a Theory of Powerful Selection of Distribution Functions", AFML-TR-71-52, March 1971.

2. Sarhan, A.E. & Greenberg, B.G.: "Contributions to Order Statistics", J. Wiley & Sons, Inc., New York, London, 1962.

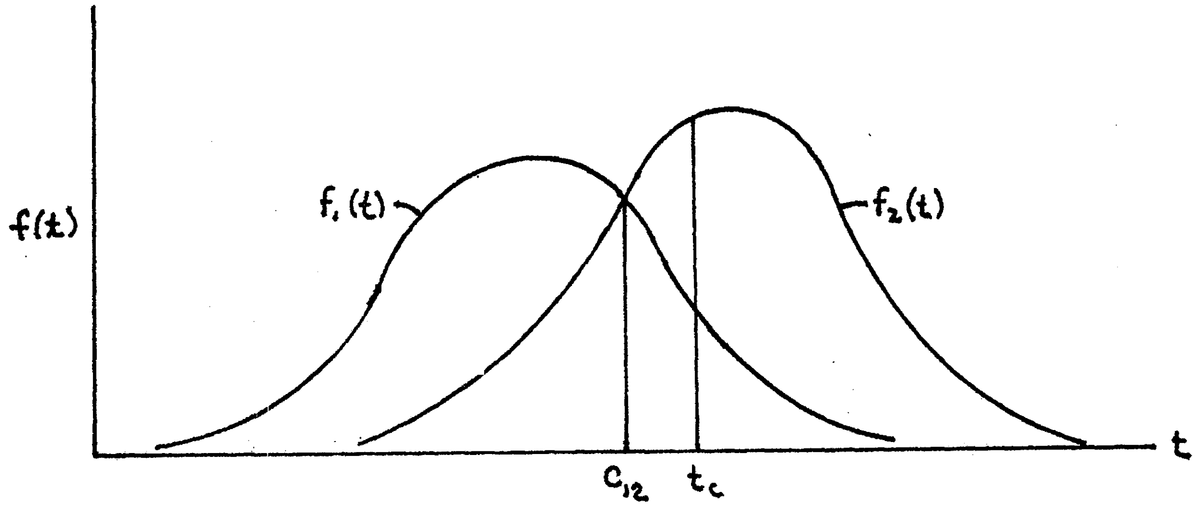

Figure 1. Selection of One of Two Hypotheses.

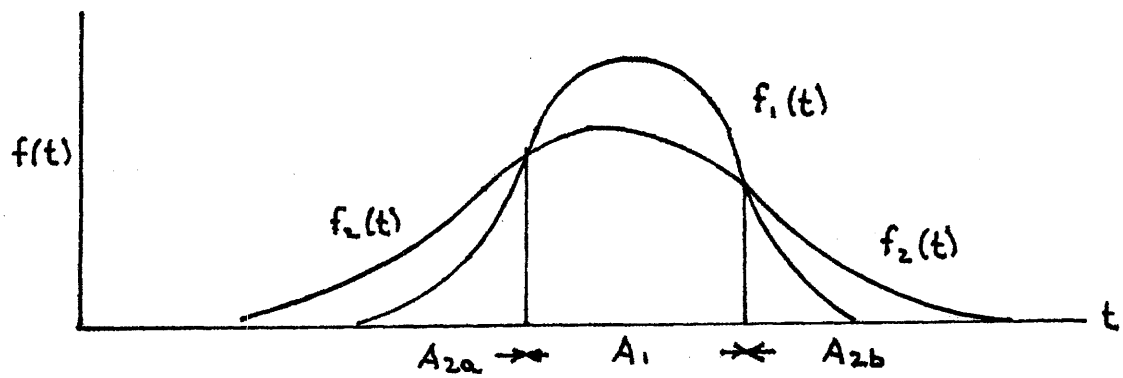

Figure 2. Double Intersections Between the Density Functions.

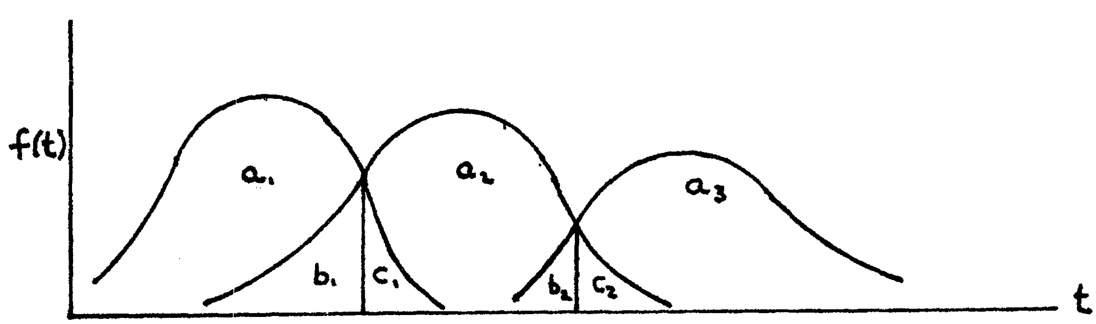

Figure 3. Selection of One of Three Hypotheses.

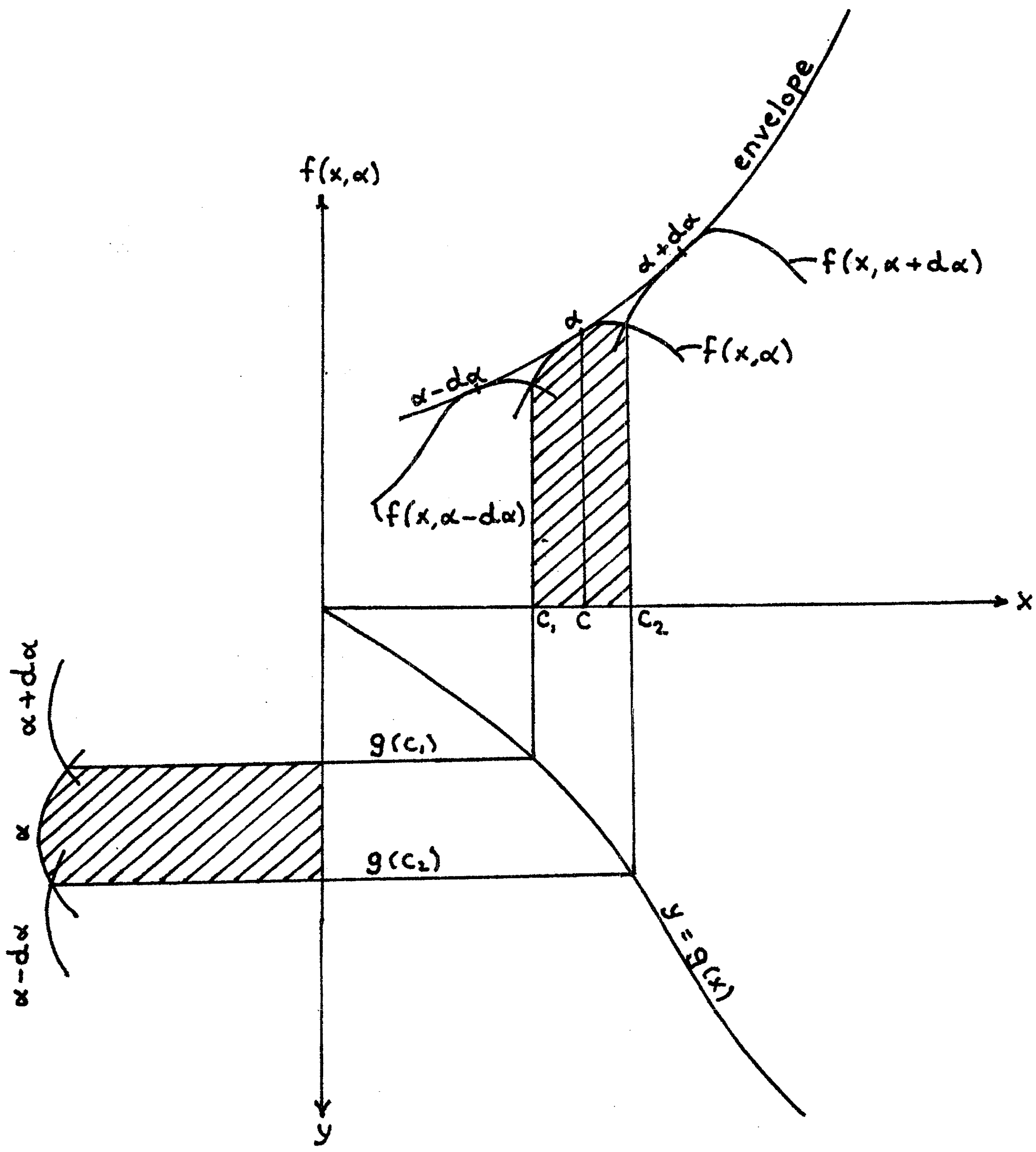

Figure 4. Transformation of the Observations.

TABLE 1. THE FUNCTION $t(u) = 1 + \ln u - u.\ln u$ $(u = z^m)$

u	t(u)	u	t(u)	u	t(u)	u	t(u)
.00	$-\infty$	0.0	$-\infty$	1.0	1.00000	2.0	0.30685
.01	-3.55912	0.1	-1.07233	1.1	0.99047	2.1	0.18387
.02	-2.83378	0.2	-0.28754	1.2	0.96354	2.2	0.05385
.03	-2.40136	0.3	0.15722	1.3	0.92129	2.3	-0.08276
.04	-2.09012	0.4	0.45023	1.4	0.86541	2.4	-0.18483
.05	-1.84594	0.5	0.65253	1.5	0.79727	2.5	-0.37912
.06	-1.64461	0.6	0.77567	1.6	0.71800	2.6	-0.52883
.07	-1.47311	0.7	0.89300	1.7	0.62856	2.7	-0.68854
.08	-1.32363	0.8	0.95537	1.8	0.52977	2.8	-0.85838
.09	-1.19125	0.9	0.98946	1.9	0.42233	2.9	-0.94404
.10	-1.07233	1.0	1.00000	2.0	0.30685	3.0	-1.19722

$t(u) = 0$ for $u_a = 0.25924$ and $u_b = 2.23893$