

MOMENTS ABOUT SMALLEST SAMPLE VALUE

WALODDI WEIBULL

*LA ROSIAZ s/LAUSANNE
SWITZERLAND*

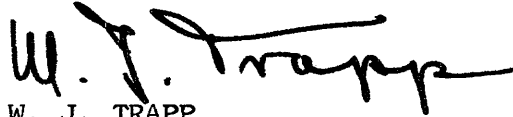
Distribution of this document is unlimited. It may be released to the Clearinghouse, Department of Commerce, for sale to the general public.

FOREWORD

This report was prepared by Prof. Dr. Wallodi Weibull, Lausanne, Switzerland, under USAF Contract No. AF 61(052)-522. The contract was initiated under Project No. 7351, "Metallic Materials, Task No. 735106, "Behavior of Metals". The contract was administered by the European Office, Office of Aerospace Research. The work was monitored by the Air Force Materials Laboratory, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, under the direction of Mr. W. J. Trapp.

This report covers work conducted during the period April 1965 to April 1966. The manuscript was released by the author in November 1966 for publication as an AFML Technical Report.

This technical report has been reviewed and is approved.

A handwritten signature in black ink, appearing to read "W. J. Trapp". The signature is fluid and cursive, with a long horizontal stroke extending to the right.

W. J. TRAPP
Chief, Strength and Dynamics Branch
Metals and Ceramics Division
Air Force Materials Laboratory

ABSTRACT

A new type of moments has been achieved by substituting in the central moments the smallest value of the sample for its mean. The new moments have the same advantage as the central moments of being independent of the location parameter but for certain values of the shape parameter they have less variance and thus are preferable for estimating purposes. The asymptotic properties of four estimators, three of them composed of the new moments and one of them of central moments have been examined. It could be concluded that for the shape parameter $\alpha \geq 0.5$ the estimator, which was composed of the first and second order moments of the new type, was by far the most efficient one. Small-sample properties of the new-moments estimators have been appraised by use of extensive Monte-Carlo studies and it could be stated that the same conclusion applies also to small and moderate sample sizes.

Distribution of this abstract is unlimited. It may be released to the Clearinghouse, Department of Commerce, for sale to the general public.

<u>Table of Contents</u>	Page
1. Introduction.	1
2. Properties of the new moments.	1
3. Use of the moments for estimating shape parameters.	3
4. Asymptotic properties of various estimators.	3
4.1 Mean and variance of the estimator R_{12} .	4
4.2 Means and variances of the estimators R_{13} and R_{23} .	5
4.3 Mean and variance of the estimator M_{23} .	6
4.4 Efficiencies of the estimators.	6
5. Small-sample properties of the estimators R_{12} and M_{23} .	7
5.1 Monte-Carlo study of the distributions of the estimators.	7
5.2 Means, medians and modes of the estimators.	7
5.3 Variances of the estimators R_{12} and M_{23} .	8
5.4 Variances of the estimates α_{12}^+ and $\alpha_{23}^{\#}$.	8
5.5 Efficiency of the estimates.	9
6. Conclusions.	10
7. References.	10
8. <u>List of tables:</u>	
No.1 Expected values of R_{12} and M_{23} for $n = \infty$.	11
No.2 Asymptotic variances of various estimates.	11
No.3 Asymptotic efficiency of various estimates.	11
No.4 Class frequencies, means and standard deviations of R_{12} for sample size $n = 5$.	12
No.5 Ditto for $n = 10$.	13
No.6 Ditto for $n = 20$.	14
No.7 Class frequencies, means and standard deviations of M_{23} for sample size $n = 5$.	15
No.8 Ditto for $n = 10$.	16
No.9 Ditto for $n = 20$.	17

<u>8. List of tables (Continued)</u>	Page
No.10 Monte-Carlo determined values of \bar{R}_{12}	18
No.11 Monte-Carlo determined values of \bar{M}_{23}	18
No.12 The statistics $\bar{R}_{12}, \check{R}_{12}$ and \tilde{R}_{12} as functions of α_0 and n .	19
No.13 The means \bar{M}_{23} as functions of α_0 and n .	20
No.14 Monte-Carlo determined values of $n.\text{Var}(R_{12})$.	21
No.15 Monte-Carlo determined values of $n.\text{Var}(M_{23})$.	21
No.16 The variances $n.\text{Var}\alpha_{12}^+$ and $n.\text{Var}\alpha_{23}^{\#}$ as functions of α_0 and n .	22
No.17 The efficiencies $\text{Eff}\alpha_{12}^+$ and $\text{Eff}\alpha_{23}^{\#}$ as functions of α_0 and n .	22
<u>9. List of figures.</u>	
No.1 Cumulative distribution of R_{12} for $n=20$.	23
No.2 Cumulative distribution of M_{23} for $n=20$.	24
No.3 The mean \bar{R}_{12} as function of α_0 and n .	25
No.4 The mean \bar{M}_{23} as function of α_0 and n .	26

Moments about Smallest Sample Value

1. Introduction

The central moments, defined by

$$m_v = \Sigma (x_i - \bar{x})^v / n \quad (1)$$

where \bar{x} denotes the mean of the sample values, has the useful property of being independent of the location parameter as easily seen by introducing the standardized variate

$$z = (x - \mu) / \beta \quad (2)$$

or

$$x = \beta z + \mu$$

Thus

$$m_v = \beta^v \Sigma (z_i - \bar{z})^v / n = \beta^v \cdot f_v(\alpha) / n \quad (3)$$

is uniquely determined by α and β .

Considering that the variances of the mean \bar{x} and of the smallest sample value x_1 from a Weibull population are

$$\text{Var}(\bar{x}) = \mu_2 / n \quad \text{and} \quad \text{Var}(x_1) = \mu_2 / n^{2\alpha} \quad (4)$$

it can be concluded that for $\alpha > 0.5$ the statistic x_1 is much more precise than \bar{x} for large sample sizes n . Thus it seems plausible that the efficiency of the moment estimators can be increased by substituting x_1 for \bar{x} in equ.(1).

In this way a new type of moments is obtained, which will be called "moments about smallest sample value", be denoted by r_v and defined by

$$r_v = \Sigma (x_i - x_1)^v / n \quad (5)$$

The properties of these moments and their usefulness for estimating purposes will now be examined.

2. Properties of the new moments

Introducing (2) into (5) we have

$$r_v = \beta^v \Sigma (z_i - z_1)^v / n = \beta^v f_{rv}(\alpha) / n \quad (6)$$

from which it can be concluded, since z is uniquely determined by α , that the moments r_v , just like the moments m_v , are independent of the location parameter μ .

It should be noted that, by definition, the first central moment

$$m_1 = \Sigma(x_i - \bar{x})/n = 0 \quad (7)$$

whereas the moment

$$r_1 = \Sigma(x_i - x_1)/n = \bar{x} - x_1 > 0 \quad (8)$$

and thus offers a new tool for estimating purposes.

From (8) it follows that the expected value of r_1

$$Er_1 = E\bar{x} - Ex_1 \quad (9)$$

and considering that

$$E\bar{x} = \mu + \beta g_1 \quad \text{and} \quad Ex_1 = \mu + \beta g_1/n^\alpha \quad (10)$$

where, using a notation proposed by Dubey,

$$g_1 = \alpha ? \quad \text{and} \quad g_v = (v\alpha) ? \quad (11)$$

the expected value of r_1 becomes

$$Er_1 = \beta g_1 (1 - n^{-\alpha}) \quad (12)$$

Further, from equ.(8)

$$\text{Var}(r_1) = \text{Var}(\bar{x}) + \text{Var}(x_1) + 2 \text{Cov}(\bar{x}, x_1) \quad (13)$$

The first two terms are given by equ.(4), while the third term includes terms of the type $E(x_i x_j)$, which can be computed by means of formulae derived by Lieblein [1].

In the same way, the expected value of Er_2 of the second moment

$$r_2 = \Sigma(x_i - x_1)^2/n = a_2 - 2x_1 \cdot a_1 + x_1^2 \quad (14)$$

can be evaluated.

The expected values and the variances of the moments r_v

take very simple forms for large sample sizes, since

$$r_v = \Sigma(x_i - x_1)^v/n = \beta^v \Sigma(z_i - z_1)^v/n \longrightarrow \beta^v \Sigma z_i^v/n \quad (15)$$

as $n \longrightarrow \infty$.

From formulae given in Scientific Report No.6 (SR 6) it then follows that

$$Er_v = \beta^v \cdot \xi_v \quad (16)$$

and

$$n \cdot \text{Var}(r_v) = \beta^{2v} (\xi_{2v} - \xi_v^2) \quad \text{and} \quad n \cdot \text{Cov}(r_v, r_w) = \beta^{v+w} (\xi_{v+w} - \xi_v \cdot \xi_w) \quad (17)$$

3. Use of the moments for estimating shape parameters

From equ.(6) it is easy to see that by use of two moments r_v of different orders, an expression can be constructed that is independent of β and thus a function of α only, for example, the expressions

$$R_{12} = r_2/r_1^2 \quad R_{13} = r_3/r_1^3 \quad R_{23} = r_3^2/r_2^3 \quad (18)$$

Each such expression can be used as an estimator of α .

Since $m_1 = 0$, the only corresponding estimator, composed of central moments, is

$$M_{23} = m_3/m_2^{3/2} \quad (19)$$

The efficiency of these estimators will now be examined. Due to the fact that the moments r_v for large sample sizes tend to the moments about origin of the standardized variate, closed expressions can in this case be deduced. For small and moderate sample sizes the Monte-Carlo Method has to be used.

4. Asymptotic properties of various estimators

If $n \longrightarrow \infty$, the estimators R tend to

$$R_{12} = a_2/a_1^2 \quad R_{13} = a_3/a_1^3 \quad R_{23} = a_3^2/a_2^3 \quad (\mu = 0) \quad (20)$$

where a_v are the moments about origin of the standardized variate.

According to a theorem stated and proved by Cramér [2,p.352] for the case of a function $H(m_v, m_w)$ of two central moments, we have, denoting by H_0, H_1 and H_2 the values assumed by the function H and its first order partial derivatives in the point $m_v = \mu_v, m_w = \mu_w$, the mean and the variance of H are given by

$$E(H) = H_0 + O(1/n) \quad (21)$$

$$\text{Var}(H) = H_1^2 \cdot \text{Var}(m_v) + 2H_1H_2 \cdot \text{Cov}(m_v, m_w) + H_2^2 \cdot \text{Var}(m_w) + O(1/n^{3/2})$$

Since these formulae are valid for any mean m of the population, including $m=0$, they are valid also for the moments a_v and r_v .

4.1 Mean and variance of the estimator R_{12} -

From equ.(20) we thus have

$$H(a_1, a_2) = a_2/a_1^2 \quad (\mu = 0) \quad (22)$$

and

$$H_0 = g_2/g_1^2 ; \quad H_1 = -2g_2/\beta g_1^3 ; \quad H_2 = 1/\beta^2 g_1^2 \quad (23)$$

and by equ.(17)

$$\begin{aligned} n \cdot \text{Var}(r_1) &= \beta^2(g_2 - g_1^2) ; \quad n \cdot \text{Cov}(r_1, r_2) = \beta^3(g_3 - g_1 g_2) ; \\ n \cdot \text{Var}(r_2) &= \beta^4(g_4 - g_2^2) \end{aligned} \quad (24)$$

which after some calculations results in

$$E(R_{12}) = g_2/g_1^2 \quad (25)$$

and

$$n \cdot \text{Var}(R_{12}) = (4g_2^3 + g_1^2 g_4 - 4g_1 g_2 g_3 - g_1^2 g_2^2)/g_1^6 \quad (26)$$

If now a value R_{12} has been computed from a random sample, the estimated value α_{12}^+ of the parameter α is obtained by equating the actual value R_{12} to the expected value $E(R_{12})$.

Hence

$$R_{12} = g_2/g_1^2 = f(\alpha_{12}^+) \quad (27)$$

Some values of the function $f(\alpha)$ are presented in Table 1.

Since $\text{Var}(R_{12}) \rightarrow 0$ as $n \rightarrow \infty$ and $R_{12} \rightarrow f(\alpha_0)$

where α_0 is the true value of the parameter, it follows that the value a_{12} computed from equ.(27) is a consistent estimate of α . Further, that the tangent curve at α_0 can be substituted for the function $f(\alpha_{12}^+)$.

Hence

$$f(\alpha_0) + (\alpha_{12}^+ - \alpha_0) \cdot f'(\alpha_0) = R_{12} \quad (28)$$

and

$$\text{Var}(\alpha_{12}^+) = \text{Var}(R_{12})/[f'(\alpha_0)]^2 \quad (29)$$

From equ.(14) of SR 7

$$d g_v / d \alpha = v \cdot g_v \psi(v \alpha) \quad (30)$$

where $\psi(\alpha)$ = the digamma function, which is tabulated, and

$$f'(\alpha) = d(g_2/g_1^2) / d\alpha = 2 g_2 [\psi(2\alpha) - \psi(\alpha)] / g_1^2 \quad (31)$$

The asymptotic efficiency thus becomes

$$n \cdot \text{AsyEff}(\alpha_{12}^+) = [g_2/g_1^2 + g_4/4g_2^2 - g_3/g_1g_2 - 1/4] / [\psi(2\alpha) - \psi(\alpha)]^2 \quad (32)$$

for $\alpha \geq 0.5$.

4.2 Means and variances of the estimators R_{13} and R_{23}

Proceeding in the same way as above the asymptotic efficiencies are found to be

$$n \cdot \text{AsyEff}(\alpha_{13}^+) = [g_2/g_1^2 + g_6/9g_3^2 - 2g_4/3g_1g_3 - 4/9] / [\psi(3\alpha) - \psi(\alpha)]^2 \quad (33)$$

and

$$n \cdot \text{AsyEff}(\alpha_{23}^+) = [g_4/9g_2^2 + g_6/9g_3^2 - g_5/3g_2g_3 - 1/36] / [\psi(3\alpha) - \psi(2\alpha)]^2 \quad (34)$$

Values of these variances are presented in Table 2.

4.3 Mean and variance of the estimator M_{23} -

In order to compare the preceding estimators with the classical estimator based on central moments, the expected value and the asymptotic variance of M_{23} have been computed for various values of α according to formulae presented by Cramér [2,p.357], e.g.,

using the notations

$$k_v = \mu_v / \alpha^v \beta^v \quad (35)$$

which values are presented in Table 3 of SR7 for $v = 2 \div 6$, we have

$$E(M_{23}) = k_3 / k_2^{3/2} \quad (36)$$

and

$$\begin{aligned} n \cdot \text{Var}(M_{23}) = & k_6 / k_2^3 + 2.25 k_3^2 k_4 / k_2^5 + 8.75 k_3^2 / k_2^3 + 9.0 + \\ & - 3k_3 k_5 / k_2^4 - 6k_4 / k_2^2 \end{aligned} \quad (37)$$

Some values of $E(M_{23})$ are presented in Table 1, and of $\text{Var}(M_{23})$ in Table 2.

4.4 Efficiencies of the estimators

The lower limit attainable of the variances of the estimate of α , denoted by $\text{Var} \hat{\alpha}$, is given in SR4 for various alternatives. Some values for the case that all three parameters are unknown are presented in Table 2.

The efficiency, as defined by R.A. Fisher, of any estimate is obtained by dividing $\text{Var} \hat{\alpha}$ by the variance of the estimate in question. Some values resulting from this procedure are presented in Table 3, from which is seen that all the estimators R are more efficient than M_{23} and that R_{12} is the most efficient of them all. It should be noted that this conclusion is valid for $\alpha \geq 0.5$ only. For $\alpha < 0.5$ the estimator M_{23} is more efficient than the estimator R .

5. Small-sample properties of the estimators R_{12} and M_{23}

5.1 Monte-Carlo study of the distributions of the estimators

An extensive Monte-Carlo study has been performed in order to determine the sampling distributions of the two estimators. By use of an IBM 7090 computer 10,000 random samples for each combination of $\alpha = 0.001, 0.01, 0.1, 0.3, 0.5, 0.7, 0.9, 1.0, 1.5, 2.0, 5.0$ and 10.0 and $n = 3, 4, 5, 10$ and 20 have been produced, from which the values of R_{12} and M_{23} have been computed and classified in 40 classes. The means and the standard deviations of the ten thousand values of the sixty combinations have at the same time been determined. The values of R_{12} are presented in Tables 4, 5 and 6 for $n = 5, 10$ and 20 , and those of M_{23} in Tables 7, 8 and 9 for $n = 5, 10$ and 20 .

Since the moments m_v and r_v tend to zero as α tends to zero, the estimators tend to the ratio $0/0$. For this reason, the data corresponding to $\alpha = 0.001$ have been omitted as being somewhat unreliable, and so also the data for $\alpha = 5.0$ and 10.0 as being of less practical interest.

The cumulative distributions of R_{12} and M_{23} have been plotted as demonstrated in Figs. 1 and 2, respectively.

5.2 Means, medians and modes of the estimators

The Monte-Carlo determined means \bar{R}_{12} and \bar{M}_{23} , computed from the 10,000 values, are presented in Tables 10 and 11, including also the values for $\alpha = 5.0$ and 10.0 .

It was found that the statistics $\bar{R}_{12}, \bar{R}_{13}$ and \bar{R}_{12} and corresponding statistics with regard to M_{23} could with excellent approximation be expressed as polynomials of the second degree in α . Fitted values and corresponding equations are presented in Tables 12 and 13.

The medians \check{R}_{12} and the modes \check{R}_{12} have been graphically evaluated from the distribution curves in Fig. 1 and the corresponding curves for $n = 5$ and $n = 10$. These values are also included in Table 12.

5.3 Variances of the estimators R_{12} and M_{23}

The Monte-Carlo determined variances $n \cdot \text{Var}(R_{12})$ and $n \cdot \text{Var}(M_{23})$, obtained by squaring the standard deviations and multiplying the squares by n , are presented in Tables 14 and 15.

5.4 Variances of the estimates α_{12}^+ and $\alpha_{23}^\#$

On the condition that the estimate α_{12}^+ is obtained by equating observed values of R_{12} to the expected value of R_{12} , as indicated by equ.(27), the variance of α_{12}^+ is computed from the variance of R_{12} by use of equ.(29). This procedure is strictly correct for $n = \infty$, but an error will be introduced for finite sample sizes. Nevertheless, this method will be applied, and the result will be used as a measure of the variance.

From the equations in Tables 12 and 13 it follows that for

$$\begin{aligned} n = 5 & \quad d\bar{R}_{12}/d\alpha = 0.2835 + 0.4280\alpha \\ n = 10 & \quad d\bar{R}_{12}/d\alpha = 0.2321 + 0.9324\alpha \end{aligned} \quad (38)$$

$$n = 20 \quad d\bar{R}_{12}/d\alpha = 0.1711 + 1.2724\alpha$$

$$n = 5 \quad d\bar{M}_{23}/d\alpha = 1.050 - 0.480\alpha$$

$$n = 10 \quad d\bar{M}_{23}/d\alpha = 2.094 - 1.152\alpha \quad (39)$$

$$n = 20 \quad d\bar{M}_{23}/d\alpha = 2.830 - 1.552\alpha$$

By use of these formulae and equ.(29) the variances of α_{12}^+ and $\alpha_{23}^\#$ have been computed. The results are presented in Table 16. It is of interest to note that for $\alpha \geq 0.5$ the variance of $\alpha_{23}^\#$ is always larger than that of α_{12}^+ .

It is obvious that this method of estimation will provide estimates with negative bias of α_{12}^+ and positive bias of $\alpha_{23}^\#$ but these biases will tend to zero with n . However, if a large set of samples are used for computing R_{12} and the mean of the values R_{12} is introduced into equ.(27), an unbiased estimate will result.

The above-mentioned method of estimation is not the only one possible. It would be quite reasonable to equate the actual value of R_{12} to the median \bar{R}_{12} . The advantage of doing so would

be that a correct median estimate will be obtained, that is, there would be 50% probability of having a value larger than and 50% probability of having a value smaller than the true value of α . Most likely, the variance of this estimate will be a little larger, but the bias a little smaller than those of the preceding method.

A third alternative of estimation would be to equate the actual value of R_{12} to the mode \tilde{R}_{12} . Such an estimate could be called the most probable estimate. To illustrate this statement by an example, suppose that from a sample of size $n=20$ a value $R_{12}=1.45$ has been computed. Knowing nothing about the true value of α , it may be assumed that there is the same probability for all values of α between 0 and 1. (In many cases these will, for logical reasons, be the limits). Then it will be found from Table 6 that, the total number of R_{12} being 7422, the probabilities of a value 1.45 coming from a population with the shape parameter α_0 will be for

α_0	=	0.1,	0.3,	0.5,	0.7,	0.9
P	=	5.7	16.8	37.1	25.6	7.9 %

The confidence limits of this estimate are easily set. Very likely, the variance of this estimate will be somewhat larger than those of the two preceding methods, but possibly the bias will be the least one.

It should be noted that the differences between the three methods will disappear with increasing sample size, since all the distributions are tending to the same normal distribution.

5.5 Efficiency of the estimates

In order to obtain the efficiency of the various estimates, the variance $\text{Var}\hat{\alpha}$ has to be divided by the respective variances. The result thus reached is presented in Table 17. It should be observed that the indicated values of the lower limit $n \cdot \text{Var}\hat{\alpha}$ is strictly valid for large sample sizes only, so another approximation has thereby been introduced. However, it is believed that the values will be sufficiently accurate for a comparison between the estimates, even if they do not give the real efficiencies, which may be better than indicated in the table.

6. Conclusions

An examination of the asymptotic properties of four estimators, three of them based on moments about the smallest sample value and one on central moments, has proved that for $\alpha \geq 0.5$ the first type of estimators is much more efficient than the second type, and that the estimator composed of the first and the second moment about smallest sample value is by far the most efficient one.

Encouraged by this result, an extensive Monte-Carlo study has been performed with the result that the same conclusion applies also to moderate and small sample sizes.

The computational work, when using any one of the moment estimators is considerably less than that required by the Maximum Likelihood Method, which, however, has the advantage of being more efficient.

7. References

1. Lieblein, J., "On moments of order statistics from the Weibull distribution." Ann.Math.Statist. 26,1955,230-233.
2. Cramér, H., Mathematical Methods of Statistics. Princeton Mathematical Series No.9, 1946.

Table 1. Expected values of R_{12} and M_{23} for $n = \infty$

α	$E(R_{12})$	$E(M_{23})$	α	$E(R_{12})$	$E(M_{23})$
0.00	1.0000	-1.1396	0.60	1.3801	0.8960
0.01	1.0002	-1.0809	0.70	1.5045	1.1604
0.10	1.0145	-0.6376	0.80	1.6480	1.4295
0.20	1.0524	-0.2541	0.90	1.8124	1.7080
0.30	1.1093	0.0687	1.00	2.0000	2.0000
0.40	1.1831	0.3586	1.50	3.3953	3.8023
0.50	1.2732	0.6311	2.00	6.0000	6.6188

Table 2. Asymptotic variances of various estimates

α	n.Var α_{12}^+	n.Var α_{13}^+	n.Var α_{23}^+	n.Var $\alpha_{23}^{\#}$	n.Var $\hat{\alpha}$
0.5	0.15574	0.16715	0.20545	0.88939	0.15198
0.6	0.23484	0.27000	0.36939	1.37268	0.21865
0.7	0.34454	0.42751	0.65061	2.04212	0.29789
0.8	0.49676	0.66876	1.12759	3.33016	0.38908
0.9	0.70731	1.03759	1.92656	5.17201	0.49242
1.0	1.00000	1.60000	3.25000	8.00000	0.60793

Table 3. Asymptotic efficiency of various estimates

α	Eff α_{12}^+	Eff α_{13}^+	Eff α_{23}^+	Eff $\alpha_{23}^{\#}$
0.5	97.59	90.92	73.97	17.09
0.6	93.19	81.06	59.25	15.94
0.7	86.46	69.68	45.78	14.59
0.8	78.32	58.18	34.50	11.68
0.9	69.62	47.46	25.56	9.52
1.0	60.79	38.00	18.71	7.60

Table 4. Class frequencies, means and standard deviations of R_{12} -
for sample size $n=5$ -

Upper class limit	The true shape parameter α_0 equal to								
	.01	.1	.3	.5	.7	.9	1.0	1.5	2.0
1.2	0	0	0	0	0	0	0	0	0
1.3	1616	1263	728	386	228	157	131	55	28
1.4	2641	2493	1891	1327	907	650	529	241	138
1.5	1735	1833	1780	1554	1134	815	729	382	202
1.6	1226	1304	1535	1418	1319	1028	910	467	277
1.7	980	1028	1185	1329	1274	1148	1051	697	442
1.8	609	669	842	1041	1063	1050	1018	675	498
1.9	369	424	566	740	882	876	824	695	468
2.0	252	303	417	550	685	784	811	687	521
2.1	184	209	279	409	547	652	680	644	500
2.2	136	145	239	295	428	518	582	620	533
2.3	74	110	146	269	348	429	476	561	523
2.4	55	61	114	171	258	352	374	558	543
2.5	33	45	75	127	203	308	292	472	532
2.6	27	28	54	78	153	210	297	368	459
2.7	14	22	36	67	120	185	204	349	362
2.8	11	16	25	73	95	136	180	285	326
2.9	7	8	23	34	63	142	144	225	290
3.0	5	9	12	34	62	99	129	210	289
3.1	7	6	9	17	60	93	121	202	231
3.2	4	6	17	22	42	62	95	178	257
3.3	3	4	1	12	28	62	69	168	229
3.4	5	4	8	13	23	55	70	155	207
3.5	2	4	4	9	21	49	57	156	188
3.6	2	2	7	5	12	37	63	124	173
3.7	1	1	2	6	17	24	38	123	195
3.8	0	1	2	5	7	17	31	114	167
3.9	0	0	0	4	7	21	28	97	156
4.0	0	0	1	2	1	15	19	108	145
ω	2	2	2	3	13	26	48	384	1121
R_{12}	1.5174	1.5443	1.6166	1.7075	1.8163	1.9395	2.0051	2.3469	2.6718
S.D.	.26665	.28020	.31659	.36469	.42678	.50075	.54039	.73342	.87908

Table 5. Class frequencies, means and standard deviations of R_{12} --
for sample size $n = 10$

Upper class limit	The true shape parameter α equal to								
	.01	.1	.3	.5	.7	.9	1.0	1.5	2.0
1.1	0	0	0	0	0	0	0	0	0
1.2	2412	1681	443	81	18	5	4	1	0
1.3	3247	3142	2333	961	301	87	65	7	1
1.4	2146	2401	2547	1955	962	429	252	38	9
1.5	1193	1364	1965	2060	1514	838	639	108	22
1.6	539	753	1196	1717	1630	1095	841	212	62
1.7	266	336	727	1212	1461	1279	1040	351	97
1.8	101	183	358	756	1191	1220	1146	441	156
1.9	51	67	205	513	865	1051	1023	545	231
2.0	21	38	95	295	617	872	920	619	285
2.1	13	17	59	188	458	711	778	614	307
2.2	7	10	35	87	308	545	656	637	356
2.3	2	4	16	56	224	455	530	619	422
2.4	1	2	12	47	128	338	421	562	434
2.5	1	1	4	29	92	243	357	497	397
2.6	0	1	2	17	69	200	257	549	414
2.7	0	0	2	8	43	134	225	404	403
2.8	0	0	0	6	31	102	152	406	400
2.9	0	0	0	6	21	84	127	371	361
3.0	0	0	0	3	23	66	115	329	388
3.1	0	0	0	2	11	60	77	285	338
3.2	0	0	0	0	10	43	71	275	349
3.3	0	0	1	0	8	22	53	252	342
3.4	0	0	0	0	7	29	45	206	273
3.5	0	0	0	0	2	11	46	184	290
3.6	0	0	0	0	1	21	22	151	235
3.7	0	0	0	0	0	13	25	128	248
3.8	0	0	0	0	2	15	16	121	217
3.9	0	0	0	0	0	7	18	119	202
4.0	0	0	0	0	0	4	20	89	192
∞	0	0	0	0	3	21	59	880	2569
\bar{R}_{12}	1.3108	1.3399	1.4245	1.5426	1.6993	1.8959	2.0083	2.6752	3.4156
S.D.	.14558	.15588	.18590	.23389	.31210	.42749	.49847	.93100	1.36031

Table 6. Class frequencies, means and standard deviations of R_{12} -
for sample size $n = 20$

Upper class limit	The true shape parameter α_0 equal to								
	.01	.1	.3	.5	.7	.9	1.0	1.5	2.0
1.0	0	0	0	0	0	0	0	0	0
1.1	799	286	5	0	0	0	0	0	0
1.2	4947	4191	1297	109	3	0	0	0	0
1.3	3009	3624	3857	1302	165	15	6	0	0
1.4	947	1356	2938	2993	921	166	69	1	0
1.5	240	420	1245	2753	1900	588	276	7	0
1.6	45	93	454	1562	2302	1137	684	25	3
1.7	11	24	147	733	1806	1534	1053	90	4
1.8	1	4	38	329	1244	1535	1274	156	12
1.9	0	1	14	138	701	1370	1320	259	43
2.0	1	1	2	42	398	1058	1191	421	48
2.1	0	0	1	24	288	793	955	479	80
2.2	0	0	1	7	124	534	799	528	142
2.3	0	0	1	4	77	376	582	622	172
2.4	0	0	0	0	22	268	453	633	233
2.5	0	0	0	1	17	194	324	659	265
2.6	0	0	0	2	7	135	238	561	294
2.7	0	0	0	1	12	96	205	581	290
2.8	0	0	0	0	5	59	136	547	354
2.9	0	0	0	0	2	49	100	460	365
3.0	0	0	0	0	3	27	90	402	360
3.1	0	0	0	0	1	17	66	399	378
3.2	0	0	0	0	0	8	45	332	373
3.3	0	0	0	0	1	9	26	300	375
3.4	0	0	0	0	0	8	25	269	362
3.5	0	0	0	0	0	2	19	238	332
3.6	0	0	0	0	0	4	16	210	289
3.7	0	0	0	0	0	5	6	193	315
3.8	0	0	0	0	0	3	4	178	294
3.9	0	0	0	0	1	0	7	121	268
4.0	0	0	0	0	0	4	6	142	241
∞	0	0	0	0	0	6	25	1198	4108
\bar{R}_{12}	1.1996	1.2274	1.3133	1.4411	1.6211	1.8605	2.0038	2.9448	4.1628
S.D.	.08691	.09347	.11207	.14486	.20908	.32129	.39863	.99530	1.80312

Table 7. Class frequencies, means and standard deviations of M_{23} -
for sample size $n=5$

Upper class limit	The true shape parameter α_0 equal to								
	.01	.1	.3	.5	.7	.9	1.0	1.5	2.0
- 2.0	16	0	0	0	0	0	0	0	0
- 1.8	1	0	0	0	0	0	0	0	0
- 1.6	5	0	0	0	0	0	0	0	0
- 1.4	124	85	44	20	15	11	10	4	2
- 1.2	525	411	179	108	61	34	26	14	10
- 1.0	615	550	348	176	100	72	58	25	11
- 0.8	740	639	392	288	176	114	96	33	21
- 0.6	837	830	595	366	269	182	150	78	31
- 0.4	998	1020	875	637	453	320	264	148	82
- 0.2	1325	1316	1199	986	780	638	595	326	234
0.0	1303	1306	1265	1104	889	723	639	412	263
0.2	911	1105	1273	1292	1141	944	867	570	384
0.4	891	1015	1226	1444	1479	1397	1349	1034	739
0.6	557	642	945	1066	1219	1309	1277	1260	1260
0.8	430	422	596	812	973	1044	1079	923	852
1.0	263	306	445	677	789	913	978	1031	842
1.2	197	206	346	516	763	894	929	1200	1105
1.4	127	121	213	404	683	985	1110	1447	1649
1.6	56	26	59	104	210	420	573	1495	2515
1.8	22	0	0	0	0	0	0	0	0
2.0	10	0	0	0	0	0	0	0	0
ω	47	0	0	0	0	0	0	0	0
\bar{M}_{23}	-.1948	-.1732	.0148	.1924	.3486	.4812	.5392	.7630	.9099
S.D.	1.07705	.61352	.60428	.60034	.59812	.59401	.59089	.56718	.53784

Table 8. Class frequencies, means and standard deviations of M_{23} —
for sample size $n=10$

Upper class limit	The true shape parameter α_0 equal to								
	.01	.1	.3	.5	.7	.9	1.0	1.5	2.0
- 2.0	123	34	3	0	0	0	0	0	0
- 1.8	131	75	4	1	0	0	0	0	0
- 1.6	235	106	9	2	1	0	0	0	0
- 1.4	334	190	37	6	2	1	0	0	0
- 1.2	514	320	71	10	2	1	1	0	0
- 1.0	775	561	165	32	10	5	2	0	0
- 0.8	947	832	303	79	20	5	6	0	0
- 0.6	1090	1055	554	201	68	28	13	5	0
- 0.4	1278	1249	890	386	161	62	46	9	0
- 0.2	1288	1378	1251	691	321	162	113	25	13
0.0	1201	1292	1503	1105	617	329	246	60	19
0.2	819	1111	1466	1399	1002	637	500	158	60
0.4	573	791	1295	1501	1259	927	796	335	139
0.6	342	487	997	1354	1434	1219	1053	553	314
0.8	197	288	676	1176	1348	1342	1269	857	498
1.0	81	115	386	853	1185	1284	1294	1008	774
1.2	35	62	185	503	912	1151	1221	1089	877
1.4	23	33	112	316	647	916	1011	1226	1057
1.6	10	10	47	184	391	665	767	1104	1167
1.8	2	8	28	117	278	461	571	789	938
2.0	0	2	11	46	177	341	433	786	809
2.2	2	0	6	28	107	255	330	696	851
2.4	0	1	0	8	48	148	218	625	935
2.6	0	0	1	2	9	58	102	561	1079
2.8	0	0	0	0	1	3	8	114	470
∞	0	0	0	0	0	0	0	0	0
\bar{M}_{23}	-.4976	-.3324	.0363	.3670	.6478	.8821	.9840	1.3768	1.6356
S.D.	.63159	.59231	.54587	.54962	.57581	.60266	.61371	.64178	.63899