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OUTLINE OF AN ALGEBRA OF STOCHASTIC QUANTITIES

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FOREWORD

This report was prepared by Prof. Dr. Waloddi Weibull, La Rosiaz, Lausanne, Switzerland under USAF Contract No. AF 61(052)-522. The contract was initiated under Project No. 7351, "Metallic Materials", Task No. 735106, "Behavior of Metals". The contract was administered by the European Office, Office of Aerospace Research. The work was monitored by the Directorate of Materials and Processes, Aeronautical Systems Division, under the direction of Mr. W. J. Trapp.

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ABSTRACT

The aim of the work reported has been to develop methods of solving random equations, that is, equations involving variates (random variables). The main difficulty of this task arises from the fact that no variate, if not degenerate, is invertible, or, algebraically expressed, even if the set V of variates is a commutative monoid under both addition and multiplication, it does not constitute a field.

For this purpose a set S of elements, called stochastic quantities (for brevity, stochastics), of which V is a subset, has been constructed with the property that it constitutes a field. This implies that there exists for every element of it an inverse element relative to both the additive and multiplicative laws of composition, and thus it will be possible to compute with the stochastics just as easily as is done with the rational numbers with respect to the four fundamental operations $+$, $-$, \cdot , $:\cdot$.

Considering a variate as a finite or infinite set of ordered pairs, denoted by $f(x) [x]$, where the first projection $f(x)$ is a real-valued, non-negative function, defined for a continuous set of values of x or for an at most denumerable set of points x_1 and interpreted as a mass density or as

discrete parts of a unit mass, respectively; and the second projection $[x]$ is anyone of the values that the variate can

take, the notation of a stochastic is $f(x) \cdot j_z^n [x]$, where $f(x)$ is a real valued, positive or negative, function and the symbol $j_z^n [x]$ is defined, for $n = 1$, by $j_z [x] = (1/dz)$

$[x] - (1/dz) [x + dz]$. Thus j_z may be interpreted as a

duplex mass, composed of two infinitely large masses $(1/dz)$ and $-(1/dz)$ located at an infinitesimal distance dz from each

other. For $n = 2$ we have $j_z^2 [x] = (1/dz^2) [x] - (2/dz^2)$

$[x + dz] + (1/dz^2) [x + 2 \cdot dz]$ and j_z^2 may be interpreted as

a triplex mass, composed of three infinitely large masses at an infinitesimal distance dz from each other, and so on for

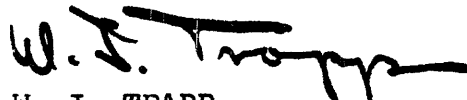
arbitrary values of n . Since, by definition, $j_z^0 [x] = 1 [x]$,

the general expression includes the variates as a special case obtained by setting $n = 0$.

From the definition above it follows, if $f(x)$ is a continuous function, that $f(x) \cdot j_z^n$ is equal to the ordinary derivative $d^n (fx)/dx^n$. Thus, j_z^n can be regarded not only as a multiplex mass but also as an operator. In the same way, j_x^{-n} , defined as the inverse of j_x^n , can be interpreted both as a mass distribution and as a repeated integration, further, the operator $d^n/dx^n = j^n$ may be defined, as is demonstrated, also for the general case that n^x is an arbitrary real number.

Since some problems leading to random equations have been presented, general properties of variates and multiplex stochastics are indicated. Based on the known laws of composition of variates, corresponding laws and some general theorems valid for multiplex stochastics have been deduced. Owing to the dual nature of the symbol j^n , simplified methods for composition and inversion of variates can be developed as is demonstrated. Finally, classification and some solutions of random equations and criteria for the existence of real roots are indicated.

This report has been reviewed and is approved.



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SECTION I. INTRODUCTION

The ordinary algebra operates with letters, which are used as symbols for real or imaginary numbers and which are composed according to elementary laws of composition thus forming equations which involve known and unknown algebraic quantities. The fundamental theorem of algebra is the proposition that every algebraic equation has a root, and its main problem is to develop methods by which the roots can be determined. A decisive property of an algebraic quantity is that it can be specified by one single number.

There is, however, another class of quantities which can take a finite or an infinite set of values, to each of which is associated a real number. The specification of such a quantity, which will be called a stochastic quantity and which includes as a special case the random variables or the variates, requires more elaborate means than a single number. Equations involving known and unknown stochastic quantities composed by use of appropriate composition laws will be called stochastic equations, including the random equations which involve variates only. The importance of such equations and the need of methods for their solution is explained by the fact that many of the quantities involved in scientific and technical problems are random variables even if they for the sake of simplicity and due to the lack of suitable methods for their solution are - frequently quite improperly - considered to be algebraic. Some examples of problems leading to random equations will be presented.

The fundamental difference between the methods for solving these two types of equations, the algebraic and the stochastic, will be illustrated by a simple example.

Denoting algebraic quantities by small letters a, b, c, \dots , if known, and by x, y, z, \dots , if arbitrary or unknown, and stochastic quantities analogously by capital letters A, B, C, \dots , and X, Y, Z, \dots , the algebraic equation

$$a = b \cdot x + c \tag{1}$$

and the stochastic equation

$$A = B \cdot X + C \tag{2}$$

will be compared.

Since the solution consists in the separation of the unknown

from the known quantities, the algebraic equation (1) is easily solved in two steps

$$a - c = b \cdot x + (c - c) = b \cdot x \quad \text{and} \quad (a - c)/b = b \cdot x/b = x \quad (3)$$

This procedure is based on the additive and multiplicative laws of composition of real numbers

$$x - x = 0 \quad \text{and} \quad x/x = 1 \quad (4)$$

These two simple laws of composition do not apply to variates or other stochastic quantities, since for any such quantity

$$X - X \neq 0 \quad \text{and} \quad X/X \neq 1 \quad (5)$$

The scope of the present investigation has been to develop methods for the solution of random equations. The leading idea was that it may be possible to find, for any given stochastic quantity X , four quantities denoted by X_+ , X_- , X_x , X_x which follow the laws of composition of variates and which satisfy the conditions

$$X + X_+ = 0 \quad X - X_- = 0 \quad X \cdot X_x = 1 \quad X/X_x = 1 \quad (6)$$

Clearly, if such quantities, which will be called inverse components, have been found, the solution of equ.(2) is just as simple as that of equ.(1) as indicated by the formal solution of equ.(2)

$$X = (A + C_+) \cdot B_x \quad (7)$$

It is easy to prove that the inverse components can never be variates and that the introduction of a new class of stochastic quantities, which will be called multiplex stochastics, is required.

For the purpose of illustration, some examples of problems leading to random equations will be presented, after which the stochastic quantities will be defined and classified. Then their general properties and the laws of their composition will be demonstrated and finally methods of solving stochastic equations will be developed.

SECTION II. SOME PROBLEMS LEADING TO STOCHASTIC EQUATIONS

The most important application of the variates is related to

the properties of elements of sets, viz., of individuals belonging to a population, say, their weights, lengths, colours, etc., taken as entities.

While some properties of a single individual, say its weight, can be specified by a single number, the same property of the individuals belonging to a population of, say, one thousand items will in the most complete form require a list giving all the individual weights. In most cases, it is, however, quite sufficient to know, for instance, that 175 items have a weight of 3 kg, 325 a weight of 2 kg, and 500 a weight of 1 kg. These figures are called the frequencies of the respective weights.

Dividing these figures by the total number 1,000 gives the relative frequencies 0.175, 0.325 and 0.500, which are also called the probabilities of the respective weights, for the reason that, if one single element is taken at random from the set, then there will be a probability of 17.5% of picking out an element having a weight of 3 kg. In cases where the set is infinitely large and the property in question may take any value within a finite or infinite interval, the probability has to be specified by a function $f(x)$, which defines the condition that the infinitesimal probability dP of drawing at random an element with a value x , belonging to the infinitesimal interval dx , is $dP = f(x)dx$. The function $f(x)$ is known as the density function (also frequency or probability function) of the variate X .

The density function provides all pertinent information on the variate. Consequently, the solution of a random equation consists in the determination of the unknown density functions from the known ones. Some examples will now be presented:

Example 1. Suppose that we have two sets, one consisting of a large number of items and the other of a large number of boxes. The weights of the boxes are represented by the variate B and those of the items by another variate C . If now the items are taken at random, one at the time, and enclosed one in each box, then the weights of the boxes with their contents is a new variate, denoted by A . We then have the random equation

$$A = B + C$$

(8)

This symbolic equation implies that, if we take at random one box from the set B , weigh it and find its weight to be b_i and one item from the set C , weigh it and find its weight to be c_i , then we can postulate that

$$a_i = b_i + c_i \quad (9)$$

is a random value from the set A, that is, the computed value a_i is a perfect substitute for the weight of a box with content actually taken at random from the set A.

This way of defining a random equation (which is not applicable to a stochastic equation in general) will be called the Monte-Carlo definition of the equation. Based on this definition, which provides also an experimental method of solving arbitrary random equations, mathematical laws of composition of variates will be deduced and extended to stochastic quantities in general.

The preceding procedure presumes that B and C are known and consequently that our equation may be written

$$X = B + C \quad (10)$$

This equation is easily solved by known methods, but let us examine another alternative

$$A = B + X \quad (11)$$

In this case, the variates A and B have been determined by weighing a sufficiently large number of boxes with and without contents and it is required to derive from this information the density function of the variate C.

If we now apply the Monte-Carlo method to this problem and take an element from the set A and find its weight to be a_i and a box from the set B and find its weight to be b_i , then it is evident that

$$x_i = a_i - b_i \quad (12)$$

does not provide a random value from the set C because, suppose that the least weight of the boxes is 2 kg and that of the items is 3 kg and that some of the boxes have a weight of 7 kg, then it may happen that we have an $a_i = 5$ kg and a $b_i = 7$ kg and thus $x_i = 5 - 7 = -2$ kg, which evidently is absurd.

The reason why the method fails in this particular case is that A and B are dependent variates, which will be denoted by

$$X = A(-)B \quad (13)$$

In an analogous way addition, multiplication, and division of dependent variates will be denoted by (+), (.), and (:), respectively.

Since equ.(12) cannot be applied without knowing the dependency between a_i and b_i , the solution of an equation even as simple as equ.(11) may be rather complicated, and still more if we take two items at random and put them into one and the same box, taken at random. The equation then takes the form

$$A = B + X + X \quad (14)$$

where the two letters X stand for two independent variates with identical density functions.

It should be noted that equ.(10), but not equ.(11), has always a real root; that is, a variate satisfying the equation. This difference between the two equations is an essential fact.

Example 2. This example is chosen to illustrate multiplication of two variates. Suppose that we have a large set of specimens. An individual specimen may have an ultimate strength s_i , cross-sectional area a_i , and specific strength b_i . Since all these values differ from item to item, we have three variates, related by

$$S = A \cdot B \quad (15)$$

Here A and B are, at least in most cases, independent variates. The variate S is easily determined from the known density functions of A and B, whereas it is much more difficult to determine B from S and A, since we have

$$X = B = S(:)A \quad (16)$$

and we do not know the dependence of S and A.

Example 3. This example is related to the important and frequently occurring problem of eliminating the influence of imperfect measuring devices used for experimental determination of variates. Let X denote the weight of a set of items which has to be determined by means of a spring balance. If the spring constant varies during the weighing procedure, it can be considered a variate denoted by B, and if the balance has a varying zero-error, this is another variate denoted by C. Then the actually observed weight A differs from the true weight X which has to be determined. Since B, C, and X

are independent of each other, the correct random equation will be

$$A = B \cdot X + C \quad (17)$$

The variates B and C can be determined by a proper calibration. This equation is typical of a large number of measuring procedures.

Example 4. A random equation relating the fatigue life N of a specimen to the imposed pulsating load S may be put in the form

$$S = (S_u - S_e)(N/b + 1)^{-a} + S_e \quad (18)$$

where S_u = the ultimate tensile strength of the specimen, S_e = its fatigue limit, a and b constants (or more correctly expressed: degenerate variates). Since S , S_u , N , a , and b can be determined by independent experiments, the variate S_e is the unknown quantity.

The solution of this equation is a rather intricate problem, complicated by the circumstance that probably S_e is dependent of S_u and possibly a and b are non-degenerate variates. This problem gave, in fact, the initial incitement to the present investigations.

SECTION III. DEFINITIONS AND CLASSIFICATION OF STOCHASTIC QUANTITIES

Let X be a set, finite or infinite, of ordered pairs $f(x)[x]$ where x is a real number and $f(x)$ a real-valued function of x . It is convenient to visualize $f(x)$ as a mass density associated to the point x on the x -axis in an n -dimensional space R_n .

Let $P(x)$ be a function which is almost everywhere equal to zero, that is, it is equal to zero except in an at most denumerable set of points x_i where it takes finite values $P_i = P(x_i)$,

Let $p(x)$ be a function almost everywhere continuous, that is, except at the discontinuity points a_i where it has a finite jump (saltus) equal to $P'_i = p(a_i)$,

Let $p'(x) = d(p(x))/dx$ be a function continuous except in the discontinuity points b_i where it has finite jumps equal to $P''_i = p'(b_i)$, etc.

Putting $f(x) = P(x_i)/dx + p(x)$, we specify the set X by

$$X = f(x)[x] = (P(x_i)/dx + p(x))[x] \quad (19)$$

or, since $P(x)$ does not exist but for $x = x_i$,

$$X = (P_i/dx)[x_i] + p(x)[x] \quad (20)$$

The function $f(x)$ will be called the density function of X and will be denoted

$$Df(X) = f(x) \quad (21)$$

The notation P_i/dx implies that the density is infinitely large at points x_i but in such a way that there is a finite mass P_i within the infinitesimal interval dx . The notation $p(x)[x]$ implies that to each value x is associated a real number $p(x)$ (interpreted as a mass density). The function $f(x)$ will be regarded as a single object which can be moved in the space R . The notation $f(x)[x]$ denotes its initial location, while the notation $f(x)[x+a]$ implies that each value (real number) $f(x)$, initially associated to x , now is associated to the point $(x+a)$ on the x -axis, that is, the object $f(x)$ has been moved a distance a on the x -axis. A move in the arbitrary z -direction an infinitesimal distance dz will be denoted by $f(x)[x+dz]$.

On the particular conditions that

$$P_i \geq 0; p(x) \geq 0 \text{ and } \sum P_i + \int_{-\infty}^{\infty} p(x)dx = 1 \quad (22)$$

the set of ordered pairs of real numbers X will be called a variate (also random variable), because then the mass may be interpreted as a probability.

When there are no such restrictions imposed on the density function, the set X will be called a real stochastic quantity or, for brevity, a real stochastic.

We will now introduce a new concept called the derivative Y of a variate or of a stochastic, which will be denoted by $D(X)/dz$ and defined by

$$Y = D(X)/dz = \lim_{\Delta z \rightarrow 0} ((f(x)[x] - f(x)[x+\Delta z])/\Delta z) \quad (23)$$

If $dz = \Delta z \rightarrow 0$ is an interval which tends to zero (but never reaches it), equ.(23) may be written, for short,

$$Y = (f(x)[x] - f(x)[x+dz])/dz$$

Taking the two terms of $f(x)$ separately, we have, since P_i is a real number and $P_i[x_i]$ may be written $P_i \cdot l[x_i]$,

$$D(P_i[x_i])dz = P_i(l[x_i] - l[x_i + dz])/dz$$

With the notation

$$j_z[x_i] = (l[x_i] - l[x_i + dz])/dz$$

we have

$$D(P_i[x_i])/dz = (P_i \cdot j_z)[x_i] \quad (24)$$

This result will, for brevity, be expressed by the statement that the derivative of a real number P_i is

$$D(P_i)/dz = P_i \cdot j_z \quad (25)$$

keeping always in mind that the real number has to be located anywhere in the space R_n .

Applying the limiting process to $P_i \cdot j_z$ results in a second derivative

$$D(P_i \cdot j_z)/dz = D^2(P_i)/dz^2 = P_i \cdot j_z^2 \quad (26)$$

and by further repetitions in general

$$D^n(P_i)/dz^n = P_i \cdot j_z^n \quad (27)$$

In the particular case $n=0$ we have

$$j_z^0[x_i] = l[x_i] \quad (28)$$

or, for short,

$$j_z^0 = 1$$

The symbol j_z^n will be defined for n equal to any real number, positive or negative, as will be demonstrated in Section 5.

The second term $p(x)[x]$ derivated becomes

$$D(p(x)[x])/dz = (p(x)[x] - p(x)[x + dz])/dz$$

and, since $p(x)$ is a real number,

$$D(p(x)[x])/dz = p(x) \cdot j_z[x] \quad (29)$$

or, for brevity,

$$D(p(x))/dz = p(x) \cdot j_z \quad (30)$$

keeping in mind that the object $p(x)$ has to be located anywhere in the space R_n and has, in the limiting process, been moved the infinitesimal distance dz in the z -direction.

The important, particular case that $z = x$ in equ.(30) and the effect of discontinuities in $p(x)$ will be discussed in Section 5.

Applying the derivation procedure to $p(x) \cdot j_z$ results in a second derivative

$$D^2(p(x))/dz^2 = p(x) \cdot j_z^2 \quad (31)$$

and by further repetitions in general

$$D^n(p(x))/dz^n = p(x) \cdot j_z^n \quad (32)$$

Combining equs.(27) and (32) we have

$$D^n(f(x))/dz^n = (P_i/dx + p(x)) \cdot j_z^n \quad (33)$$

where $f(x)$ stands for $f(x)[x]$ and j_z^n for $j_z^n[x_i]$ and $j_z^n[x]$, respectively.

Just as P_i is a real number which may be interpreted as a concentrated mass and $p(x)$ is another real number which may be interpreted as a mass density, $P_i \cdot j_z^n$ and $p(x) \cdot j_z^n$ may be called multiplex numbers and interpreted as a multiplex mass and a multiplex mass density associated to the points x on the straight x -line.

The set of ordered pairs $X \cdot j_z^n = f(x) \cdot j_z^n[x]$, where $f(x)$ is a real-valued function will be called a multiplex stochastic.

The stochastic quantities may thus be classified into: Real stochastics, including as a particular case the variates, and the multiplex stochastics. Each of these classes may be subdivided into discrete, continuous, and mixed stochastics.

In the particular case that the stochastic quantity consists of one single ordered pair, it will be called degenerate (degenerate variate, degenerate real stochastic, degenerate multiplex stochastic).

According to the conditions (22), a degenerate variate must be denoted by $1[x_i]$, and a degenerate stochastic by $k \cdot j^n[x_i]$ where k is a real number, positive or negative. The case $n=0$ corresponds to the degenerate real stochastic $k[x_i]$, where $k \neq 1$.

It is evident that the derivative of a variate can never be a variate but sometimes a real stochastic.

SECTION IV. GENERAL PROPERTIES OF VARIATES

4.1 Degenerate and Discrete Variates

A degenerate variate X is defined by the condition that it takes only one single value x_i , which implies that there is 100% probability that it takes this value, or that the mass distribution consists of a unit mass located at the point x_i . It will be denoted

$$X = (1[x_i]) \quad (34)$$

The particular case that $x_i = 0$ will sometimes be denoted by $X=0$. This notation does not imply that X disappears, and should correctly be denoted by $X = (1[0])$.

Comparing the two degenerate stochastics $(1[0])$ and $(0[1])$, the first notation implies that the probability of the value 0 is 100%, while the second one implies that there is no probability at all, that the real stochastic takes the value 1 and so far no other value, so this notation actually implies that the quantity disappears.

In the same way, $X=1$ should correctly be written $X=1[1]$, but the notations $X=0$ and $X=1$ will be accepted, when no confusion can arise.

The most simple non-degenerate variate is that one which takes two values, x_1 and x_2 with the probabilities P_1 and $P_2 = 1 - P_1$ respectively. It will be denoted by

$$X = (P_1[x_1] + P_2[x_2]) \quad (35)$$

Care must be taken to distinguish this notation from

$$X = (P_1[x_1]) + (P_2[x_2]) \quad (36)$$

which implies the sum of two degenerate stochastics.

If $x_2 \rightarrow x_1$, then $X \rightarrow (1[x_1])$, that is, to a degenerate variate, which may, for short, be written $X = x_1$.

4.2 Continuous Variates

If the density function $f(x)$ does not include any discrete, infinitely large values, it will be denoted by $p(x)$. The mass within an infinitesimal interval dx then represents the probability dP of obtaining, at random, a value belonging to this interval.

Thus

$$dP = p(x) dx \quad (37)$$

In the same way, the mass content of the interval (a,b) , corresponding to the probability P_{ab} of finding, at random, a value belonging to this interval, is

$$P_{ab} = \int_a^b p(x) dx \quad (38)$$

The function $F(x)$ defined by

$$F(x) = \int_{-\infty}^x p(x) dx \quad (39)$$

is equal to the probability that X takes a value equal to or less than x , denoted $\text{Prob}(X \leq x)$. This function is known as the cumulative distribution function and will be denoted by

$$\text{cdf}(X) = F(x) \quad (40)$$

4.3 Bounded Variates

The variates may, from another aspect, be classified into bounded and unbounded variates. The importance of the bounded variates is due to the fact that all properties of concrete objects, represented by a variate, for instance, length, weight, material strength, fatigue life, times, etc., are bounded (as never taking negative values), even if they are frequently assumed to be normal variates, which have no finite bounds.

In analogy with the notation x^+ , known as the positive part of x and defined by

$$x^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases} \quad (41)$$

we will use the notation

$$(x - b)^{+a} = \begin{cases} (x - b)^a & \text{if } x - b \geq 0 \\ 0 & \text{if } x - b \leq 0 \end{cases} \quad (42)$$

where $a \geq 0$.

A variate bounded from below has a density function which is equal to zero for $x - b \leq 0$. The value b is called the lower bound. A variate bounded from above has a density function which is zero for $b - x \leq 0$. The value b is called the upper bound of the variate.

The essential difference between bounded and unbounded continuous variates consists in the property that the density function of a bounded variate has at least one discontinuity and consequently higher multiplex derivatives, while that of an unbounded variate has no discontinuities at all.

Two classes of bounded variates of practical importance will now be examined: The first class includes all variates having a density function which can be developed into a series of the form

$$f(x) = \sum k_a (x - b)^{+a} \quad (43)$$

where the real number $a \geq 0$. It should be noted that this function and its derivatives have no discontinuities except possibly at the lower bound b .

As an example of such a density function we will take

$$f(x) = m \cdot x^{+(m-1)} \cdot e^{-x^+m} \quad (m \geq 1) \quad (44)$$

which may be developed into the series

$$f(x) = m \cdot \sum (-1)^n \cdot x^{+(n \cdot m + m - 1)} / n! \quad (45)$$

where the integer $n \geq 0$.