Problem Of The Month
September 2001---Weibull Problem #2 For Size Distribution of Fly Ash

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Waloddi Weibull's original 1951 paper, was published by ASME . This paper told about a distribution which became know as the Weibull distribution—a widely used tool in reliability circles, The ASME paper presented seven problems from widely different fields to show applicability and versatile for solving problems.

This problem of the month discusses his **second example** for size distribution of fly ash. A similar particle size type problem is discussed in the December 2000 Problem Of The Month at http://www.barringer1.com/dec00prb.htm for pulverized coal samples.

Weibull's seven problems were solved the hard way by slide rule, hand-crank calculators, graph papers, and the application of plenty of brainpower. The wide variety of problems were:

1. Yield strength of a Bofors steel (Bofors was a European producer of steel). See a discussion of this problem at http://www.barringer1.com/aug01prb.htm .
2. Size distribution of fly ash. **←This problem is described below.**
3. Fiber strength of Indian cotton.
4. Length of cytoidea (Worm length for ancient sedimentary deposits).
5. Fatigue life of a St-37 steel
6. Statures for adult males, born in the British Isles
7. Breadth of beans of Phaseolux Vulgaris.

Weibull used these examples to demonstrate the new distribution "may sometimes render good service".

Details of the Weibull distribution are described in the ASME Journal of Applied Mechanics, September 1951, pages 293-297 along with an important discussion published June 1952 on pages 233-234. For a copy of this paper see the PDF file at http://www.barringer1.com/wa_files/Weibull's-ASME-Paper-1951.pdf . Remember your dues to ASME continue to sponsor excellent papers like this and merit your support for joining ASME .

Table 1: Fly Ash Data			
(From Table 2 Of Weibull's Original 1951Paper)			
(x= particle diameter in 20 microns)			
Plot Point	**Coded Size**	**Expected Values**	**Observed Values**
n_i	**x**	**n**	**n**
1	**2**	3	**3**
2	**3**	14	**14**
3	**4**	34	**34**
4	**5**	62	**56**
5	**6**	92	**85**
6	**7**	122	**126**
7	**8**	150	**150**
8	**9**	172	**175**
9	**10**	188	**188**
10	**11**	199	**197**
11	**12**	205	**202**
12	**13**	209	**208**
13	**14**	211	**211 = N**

Weibull's second example is from "Micromerities", by J. M. Dalla Valle, Pitman Publishing Corporation, New York, NY, 1948, p.57, Fig. 2. The data is listed in Table 1. The raw data was used to prepare Table 2.

Data in Table 2 will be used for making a plot the way that Weibull would have done the task assuming the use of mean rank plotting positions (Weibull's paper is mute on what tool he used for the key plotting position feature). When mean ranks are used, we can reproduce Weibull's slopes, when Benard's median ranks (i-0.3)/(N+0.4) or Hazen plotting position ((i-0.5)/N are used, they do not produce Weibull's slopes.

Table 2: Uncorrected Fly Ash Data For Weibull Plot
(From Table 2 Of Weibull's Original 1951Paper prepared for first Weibull plot)
(x= particle diameter in 20 microns)

Plot Point	Coded Size	$\log_{10}(x)$ For X-axis	Observed Values	Cum p = $n_i/(N+1)$	1/(1-p)	$\log_{10}(\log_{10}(1/(1-p))$ For Y-axis
	x		n			
1	2	0.30103	3	0.0141	1.0143	-2.21040
2	3	0.47712	14	0.0657	1.0704	-1.52979
3	4	0.60206	34	0.1596	1.1899	-1.12190
4	5	0.69897	56	0.2629	1.3567	-0.87785
5	6	0.77815	85	0.3991	1.6641	-0.65527
6	7	0.84510	126	0.5915	2.4483	-0.41021
7	8	0.90309	150	0.7042	3.3810	-0.27651
8	9	0.95424	175	0.8216	5.6053	-0.12575
9	10	1.00000	188	0.8826	8.5200	-0.03131
10	11	1.04139	197	0.9249	13.3125	0.05087
11	12	1.07918	202	0.9484	19.3636	0.10957
12	13	1.11394	208	0.9765	42.6000	0.21203
13	14	1.14613	211 = N	0.9906	106.5000	0.30693

The data for the X-axis is simply the $\log_{10}$ of the coded size. Please note the X-axis information is in rank order of for the fly ash "screen size". The Y-axis is more complicated. Details are shown in the spreadsheet of Table 2 using the information from the observed values, which are cumulative. The Y-axis first plot position will be 3/(211+1) = 0.0141 or 1.41%, the second plot position is 14/(211+1) = 0.0657 = 6.57%, and so forth..

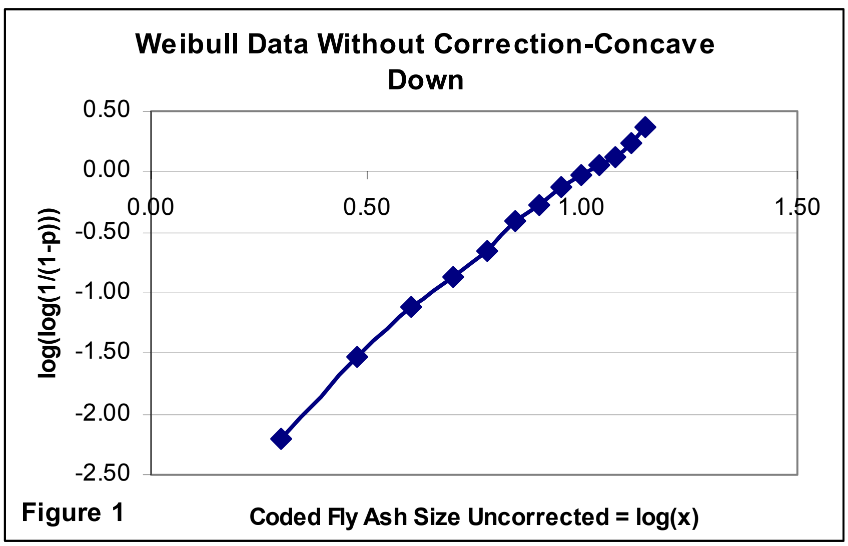


Figure 1

Table 2's <u>raw</u> data in columns labeled for the X-Y data is plotted in Figure 1. The <u>raw</u> data has a concave downward appearance and not the desired straight-line appearance. This is a clue that a correction factor, Weibull's x_u, must be subtracted from the raw data to straighten the concave downward curve into a straight line. Please note that if the correction is too great, the resulting line becomes concave upward. The process of subtraction is an iterative effort to search for the best (largest) coefficient of determination (r^2).

The x_u amount to subtract, in theory, is a straightforward task. However, the small datasets of less than 20 data points can leave you in a fool's paradise with good r^2 values and the feeling that all is right with the world when in fact the results may be suspicious. The uncorrected data for 14 X-data points and 211 Y-data points shown in Figure 1 has a straight-line r^2 =0.993 even though your eye tells you the data makes a trend line which is concave downward.

At the time of Weibull's paper, of course computers were not available. The task of iterations was a trial/error effort accomplished by slide rule, log tables, and hand cranked calculators. Thus the number of iterations was few. Today with out 1000^+Mhz computers, we can make more iterations in 10 seconds with our fast personal computers than Weibull could accomplish in a life time—this infers we can get better results today than just 50 years ago.

Table 3: Data For X-axis Plot		
Coded Size	where x_u = 1 to achieve "x" values in Weibull's Fig 2	
x	**$x-x_u$**	**$log_{10}(x-x_u)$**
2	1	**0.00000**
3	2	**0.30103**
4	3	**0.47712**
5	4	**0.60206**
6	5	**0.69897**
7	6	**0.77815**
8	7	**0.84510**
9	8	**0.90309**
10	9	**0.95424**
11	10	**1.00000**
12	11	**1.04139**
13	12	**1.07918**
14	13	**1.11394**

Table 3 shows the x_u correction to match "X" values in Weibull's Figure 2. Note that the data was probably contrived because the $x-x_u$ steps are so uniform—that does not make it bad data for illustrating the methodology. Mother nature rarely function so smoothly as to provide the $x - x_u$ order shown in the table for the corrected the plotted values in Figure 2 shown to the right. Compare the X-axis values in the right hand column of Table 3 with Weibull's Figure 2 to verify the results are correct.

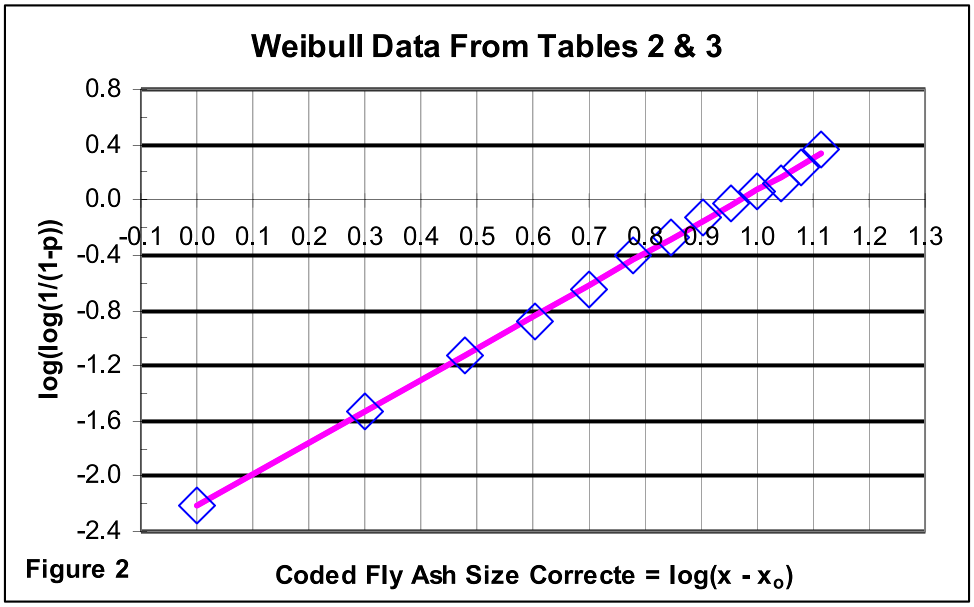


Figure 2 uses the Y-axis values from the right hand column of Table 2 and the right hand column of Table 3 to form plotting coordinates, e.g., (0.00000,-2.21040). The resulting plot of corrected data, removes the concave downward curvature, and produces a pretty good good straight line observed in Figure 2.

Regression data (regressing X onto Y) for Figure 2 shows r^2 =0.993, line slope m = 2.2879, with a Y-intercept at X= 0 of –2.2176. So the equation of the trend line in Figure 2 is Y = 2.2879*X – 2.2176. Where the trend line crosses the axis at Y = 0 defines the characteristic value for the coded fly ash as corrected for curvature in the x_u régime. In this case the X-value is 2.2176/2.2879 = 0.9693 (remember this is a transformed value in the x_u régime). The 0.9693 value calculated is consistent with the value observed from Weibull's paper in his Figure 2. The antilog of 0.9693 is $10^{0.9693}$ = 9.3175 in the x_u régime and the characteristic value in coded form, in the scale recorded is 9.3175 + 1 = 10.3175.

Using the regression line, work the problem backward to find the expected (predicted) results in Figure 1 for a comparison in Table 4. The first 3 columns of Table 4 are the same as shown in Table 3 while the remained of the data is calculated from the regression line.

Table 4: Back Calculated Results From Regression								
Coded Size	where x_u = 1 to achieve "x" values							
x	$x-x_u$	$\log_{10}(x-x_u)$	Regressed Y-values for $\log_{10}(\log_{10}(1/(1-p))$	Solve For 1/(1-p)	Solve For Cum p = $n_i/(N+1)$	Solve For Expected (predicted) values	Weibull's Expected Values	Observed Values
2	1	**0.00000**	-2.2176	1.0140	0.0139	3	3	**3**
3	2	**0.30103**	-1.5289	1.0705	0.0659	14	14	**14**
4	3	**0.47712**	-1.1260	1.1880	0.1583	34	34	**34**
5	4	**0.60206**	-0.8401	1.3947	0.2830	60	62	**56**
6	5	**0.69897**	-0.6184	1.7408	0.4256	90	92	**85**
7	6	**0.77815**	-0.4373	2.3194	0.5688	121	122	**126**
8	7	**0.84510**	-0.2841	3.3104	0.6979	148	150	**150**
9	8	**0.90309**	-0.1514	5.0773	0.8030	170	172	**175**
10	9	**0.95424**	-0.0344	8.3921	0.8808	187	188	**188**
11	10	**1.00000**	0.0703	14.9868	0.9333	198	199	**197**
12	11	**1.04139**	0.1650	28.9858	0.9655	205	205	**202**
13	12	**1.07918**	0.2515	60.8506	0.9836	209	209	**208**
14	13	**1.11394**	0.3310	138.9469	0.9928	210	211	**211 = N**

So the solved for expected values is very close (but not exactly—but then Weibull didn't have spreadsheets with firm rules for round off conditions and the humans could, perhaps, influence the end results) Weibull's values. This shows the methodology. Remember in the days before computers, you had to know many details that today we forget about when we use software. Of course you've got to have the smart guys to write the software, advance the technology, and validate the results. With software and all the complications hidden below the surface, we can now enjoy the fruits of employing less skilled engineers to operate the software.

Weibull's paper shows the distribution as a three-parameter situation. Weibull's x_u = 30 microns, x_o = 128 microns, m = 2.2883 where his equation is $F(t) = 1\text{-}e\text{^}\{(x - x_u)/x_0\}\text{^}m$. It's not so clear how he achieved his reported x_u and x_o although we can duplicate his line slope "m".

Dr. Abernethy describes the three-parameter distribution is described by in The New Weibull Handbook. In modern parlance the shift of the origin is known as t_0 and Weibull identified it as x_u. Also in modern parlance, the shape factor slope of the distribution is referred to a β whereas Weibull labeled it as m. Likewise modern parlance refers to the characteristic value as η compared to Weibull designation as x_o. Why the nomenclature changed during the past 50 years is lost in antiquity.

Today, we would see the 211 data points as data in a frequency table would appear as (screen_size * occurrences):
2*3
3*11
4*20
5*22
6*29
7*41

8*24
9*25
10*13
11*9
12*5
13*6
14*3
You can copy this data from this web page and paste it directly into WinSMITH Weibull. Select the "Method" icon, mean ranks and choose "Inspection" to process the stacks of data. Also from the main menu choose the three parameter Weibull equation and set t_0 = 1. You will find η = 6.47 (in the t_0 régime) β = 2.291, r^2 = 0.998. Click on the three-parameter icon and select the X-axis in the scale as recorded and you can see the curved Weibull plot and notice that the η = 7.47 (in the real régime) with β = 2.291, r^2 = 0.998

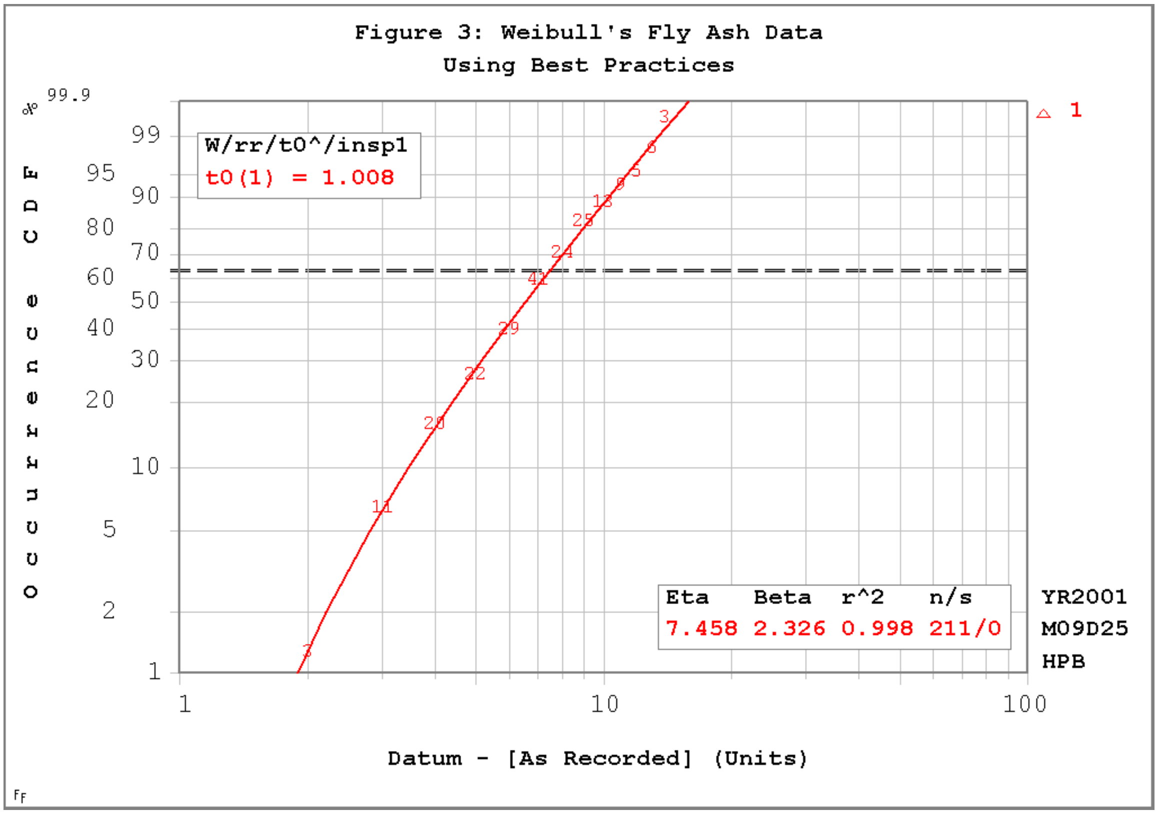
If you allow WinSMITH Weibull to find the optimum t_0, you will see t_0 = 0.9438, η = 7.487 (in the real régime), β = 2.33, r^2 = 0.998 (of course this is a rounded number).

If you're using the **demonstration version** of WinSMITH Weibull, you'll get a correct analysis but *the input data will not be exact,* as the no-cost demo version will randomize input data. For authentic results use the full strength software.

Contrast the [Copy], [Paste] effort with modern software to the large effort required in 1951 to work out the problem the hard way!

Consider the results in Figure 3 for the "right" answer based on 50 more years of experience that Weibull did not have. In this case, WinSMITH Weibull is allowed to find the value of t_0 using Benard's median rank plotting position (which in later years Weibull adopted), the inspection option for stacked data, and the results are t_0 = 1.008, η = 7.458 (in the real régime), β = 2.326, r^2 = 0.998.

Notice that Figure 3 uses median ranks as best practice, regressing X onto Y, using the inspection option for course data and finds the correct **t0 value is 1.008**.



Comments:

Refer to the caveats on the **Problem Of The Month Page** about the limitations of the following solution. Maybe you have a better idea on how to solve the problem. Maybe you find where I've screwed-up the solution and you can point out my errors as you check my calculations. E-mail your comments, criticism, and corrections to: Paul Barringer by clicking here.

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