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Classification and Analysis of Weibull Mixtures

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ABSTRACT

A bathtub equation can be used to model data that exhibits infant mortality, chance failures, and wear out. This technique allows for the simultaneous solution of equation parameters affecting the product's life. The bathtub equation treats a portion of the population as a competing risk mixture. This allows total failure of the infant mortality population without causing complete failure of the entire population. Chance and wear out failures are included by using a compound competing risk mixture.

INTRODUCTION

Reliability theory defines three typical classes in a product life cycle. The infant mortality portion has a declining hazard rate. The chance failure portion has a relatively constant hazard rate, and the wear out portion has an increasing hazard rate. Plotting the hazard rate over time results in a bathtub shaped curve. Most failure distribution models only address one or two of these classes at a time.

Many failure analysis problems include mixes of various failure modes. While it is always preferable to separate the failure modes and analyze them individually, sometimes that is not practical or possible. For example, consider a lip seal in a hydraulic cylinder that starts to leak due to particulate damage. The particulate is washed away, so it is unknown if failure occurred due to wear (chance failure particulate) or was assembled with the particulate (infant mortality particulate).

A product may have another failure mode (such as wear out) with a steeper Weibull slope. If possible, the failure modes should be examined separately. When a ball bearing wears out, it is often not noticed until the cage separates. Is the cause of the failure a load spike (chance failure), or bearing fatigue (wear out)? Cause determination may not be possible due to damage to remaining pieces after the failure.

In this paper, a method will be presented that will allow for analysis and prediction of failure rates where infant mortality, chance, and wear out failures are combined. The equations can be used when one, two, or all of the three failure classes exist.

EQUATION DEVELOPMENT

For a given failure distribution, there is an instant probability of failure $f(t)$ and a cumulative probability of failure $F(t)$, where $F(t)$ is the integral of $f(t)$. $F(t)$ may be Weibull, Normal, Log-Normal, or another distribution. The following mixture classes are the most common:

- Simple Mixtures
- Competing Risk (homogeneous population)
- Competing Risk Mixture
- Perpetual Survivors (special case of the above)
- Compound Competing Risk Mixture

SIMPLE MIXTURES – A mixture (or batch effect) occurs when there are two different sub-populations in the analysis. One sub-population may be physically different (a batch that missed a heat treating operation), or a sub-population may be exposed to a different environment (thermal issues with trucks sold in Alaska).

In their book on electrical burn in, Jensen and Peterson split the population into two groups, u and $(1 - u)$, where u is the portion of the total population that can be classified as infant mortality population[1]. This allows the infant mortality portion to fail completely without causing the entire population to completely fail. The failure rate can be expressed as:

$$F = uF_i + (1 - u)F_r \quad (1)$$

where the first term is the infant mortality population and the second term is the remaining population. Kececioglu [2] has expanded this equation to include multiple sub-populations.

The simple mixture model assumes that each population has its own failure mode and is not subject to failure mode(s) of the other sub-population(s). As shown later, this may be a reasonable assumption in many cases.

COMPETING RISK – When a uniform population is subject to two failure modes, there is competing risk. The failure mode can be expressed as:

$$F = 1 - (1-F_c)(1-F_w) \quad (2)$$

or $F = 1 - R_c R_w$ (3)

This would be a suitable model for a tire that has the chance failures of punctures (F_c), along with the wear out failures (F_w). The fact that a tire is worn out does not make it immune to puncture. The reliability of the tire will be the product of the reliability of all of its failure modes, and $R(t)=1- F(t)$, where R is the reliability at time t.

Many "mixture" problems are some sort of competing risk. From a practical standpoint, usually one failure mode is so dominant that the other risks cannot be calculated and can be safely neglected. For example, exposure of a tire to sunlight and ozone can result in deterioration of the rubber, causing a flat. This is not a concern to the sales executive who drives 160,000 km / year, but is very important to the motor home owner who drives 2000 km /year. In the first case, the tire will wear out well before the ozone can do any significant damage. But for the motor home, sunlight degradation may be the primary failure mode due to the low annual mileage.

Another example is the typical test-and-fix product development cycle. Once the dominant failure mode has been designed out, a new failure mode appears. Parts from the first design failed from the first failure mode before any significant probability of the second failure mode.

COMPETING RISK MIXTURES – A more realistic model for mixtures is to divide the population into two or more subpopulations. Each subpopulation is subject to failure modes that affect the entire population, as well as failure modes that are unique to that subpopulation.

Consider an automobile tire. It may be made out of round, causing vibration. This defect does not preclude getting a puncture while driving the car to the dealer to have the tire replaced. By combining EQ (1) and (2), the general equation for a mixture with competing risk is:

$$F = u[1 - (1-F_i)(1-F_r)] + (1 - u)F_r \quad (4)$$

where the infant mortality portion has been modified to account for the chance failures that occur to the remainder of the population.

Considering that $F_i = 1 - (1- F_i)$, compare EQ (1) and (4). If the infant mortality failure rate is significantly inferior to the failure rate of the remaining population, there will not be a significant difference in the results using either equation. For example, at time t, if F_i = 99%, F_r = 1%, and u = 40%, EQ(1) and (4) evaluate to 40.200% and 40.204% respectively. In many cases, using EQ (1) for a competing risk mixture will result in adequate (but technically inaccurate) results. Data from Examples 1 and 3 in this paper were analyzed using a simple mixture and using a competing risk mixture. There was no significant difference.

PERPETUAL SURVIVORS – Sometimes there are early failures, but no apparent chance or wear out failures. These parts are called perpetual survivors. They are not at risk of failure in their environment. During testing, it may not be clear if the parts should be tested longer to failure or if they should be suspended. Geoff Cole of Rolls Royce has developed a method [3] where the slopes of Weibull fits using rank regression and maximum likelihood are compared. This can be used to determine that there is a mixture.

COMPOUND COMPETING RISK MIXTURE OR GENERAL BATHTUB EQUATION – Failure of a population may have one or more distributions. For example, in an automobile tire, one would expect different failure distributions for punctures and wear out. Expanding F_r in EQ (4) to include chance and wear out failures results in the general bathtub equation:

$$F = u[1 - (1-F_i)(1-F_c)(1-F_w)] + (1 - u)[1 - (1-F_c)(1-F_w)] \quad (5)$$

F_i refers to infant mortality. F_c refers to chance failures. F_w refers to wear out. The first subpopulation, u[...], is subject to failure due to infant mortality, chance, and wear out. The second subpopulation, (1 – u)[...], is only subject to chance and wear out failures.

Note that EQ (5) could be expanded for additional subpopulations or failure modes.

If a reliability function, rather than a failure function, is used to describe each of the hazards, the equation becomes:

$$F = u[1 - R_i R_c R_w] + (1 - u) [1 - R_c R_w] \quad (6)$$

Note that these equations are not dependent on the distribution type, which may be Normal, Log-Normal, or Weibull. The next section shows the application of EQ (6) to Weibull distributions.

WEIBATH EQUATION DEVELOPMENT

WEIBULL EQUATION – The Weibull equation for cumulative failure rate is:

$$F(t) = 1 - \mathbf{exp}[- (t/\eta)\hat{}\beta] \quad (7)$$

while the Weibull reliability is:

$$R = 1 - F(t) = \mathbf{exp}[- (t/\eta)\hat{}\beta] \quad (8)$$

WEIBATH (WEIbull+BATHtub) EQUATION – Substituting the Weibull equation EQ(8) into the Bathtub equation EQ(6), results in a WeiBath (Weibull + Bathtub) distribution. The WeiBath failure distribution for a compound competing risk distribution is:

$$F = u\{1 - \mathbf{exp}[- (t/h)^b - (t/\eta)^\beta - (t/H)^B]\}$$

$$+ (1 - u) \{1 - \mathbf{exp}[- (t/\eta)^\beta - (t/H)^B]\} \quad (9)$$

or, by rearranging, the WeiBath equation:

$$F = 1 - (u)\mathbf{exp}[-(t/h)^b - (t/\eta)^\beta - (t/H)^B]$$

$$- (1 - u)\mathbf{exp}[-(t/\eta)^\beta - (t/H)^B] \quad (10)$$

b and *h* refer the slope and characteristic life of the infant mortality portion. β and η refer to the chance failures. *B* and *H* refer to wear out. The equation has been simplified by combining the exponentials. ($\mathbf{e}^a \bullet \mathbf{e}^b = \mathbf{e}^{a+b}$) EQ (10) could be modified with a t_o correction to account for minimum lives, but this is beyond the scope of this paper.

The first portion of the equation deals with an infant mortality subpopulation that is also exposed to chance failure and wear out. In the tire example, a tire may be out of round, but the tire may not be replaced before it wears out. The defective tire may also have a chance puncture failure.

The second portion of the equation shows a competing risk population.

This model can be used to fit a broad variety of Weibull distributions.

Setting $u = 0$ results in a competing risk equation. This is the floor and tail of the bathtub curve.

Setting $H = \infty$ results in a competing risk mixture or a no wear-out Weibull. The bathtub curve floor continues forever.

SOLUTION METHODS

RANK REGRESSION – While the Coefficient of Determination, r^2, is normally defined for linear regression, one definition[4] of r^2 is:

$$r^2 = 1 - \text{[unexplained variation] / [total variation]}$$
$$= 1 - [\Sigma(Y_c - Y)^2] / [\Sigma Y^2 - (\Sigma Y)^2/n] \quad (11)$$

In a standard linear regression, the unexplained variation is minimized by setting the derivative of $\Sigma(Y_c - Y)^2$ equal to zero and solving analytically (least squares method). An analytical solution for the Weibull bathtub curve is difficult, so the solution is best solved using numerical techniques. The *Y* axis variable, Y_c, is $\mathbf{ln}(-\mathbf{ln}(1-F(t)))$ for plotting Weibull data. Median ranks can be used for $F(t)$. The *X* axis will be $\mathbf{ln}(t)$.

Since there is more uncertainty in the time to failure than in the ranks used, Weibull analysis programs regress time against the ranks (*X* on *Y*) rather than the traditional *Y*-axis on *X*-axis approach. Care must be taken with these models to ensure they are analyzed properly for the conditions of the data. *Y* on *X* can be used for certain types of warranty or inspection data. In these cases, the actual failure rate, rather than the median rank, is used.

This adds considerable difficulty to the iteration algorithm, because for each iteration of the parameters, the inverse of bathtub curve has to be solved to determine *t*.

There are alternative criteria that can be chosen for fitting a model. Alan Townsend created the early BiWeibull competing risk models and numerical solution methods [3]. He used a modified coefficient of determination (r^2) that accounted for the number of parameters.

Levenberg-Marquardt Iteration – Some computer libraries include a non-linear equation solver based on the algorithms developed by Levenberg and Marquardt. [7,8,9] These subroutines will solve a WeiBath equation in a matter of seconds, compared to hours for Newton-Raphson. If $E = \Sigma(Y_c - Y)^2$, then the partial derivatives of *E* with respect to each variable are calculated. The Jacobian (derivatives) must be calculated to use this method.

Initial Estimates – In order to solve *E*, an initial estimate of the solution must be made. Kececioglu describes a graphical method in his book. Jiang and Murthy [5] wrote a paper on estimating parameters for mixture and competing risk models, but they do not consider competing risk mixture.

A crude but sufficient initial estimate for numerical iteration can be generated as follows:

The subpopulation portion, *u*, can be estimated by plotting the data. There will usually be a knee in the curve where the slope decreases sharply. The failure rate at this point (e.g., 20%) should be used for an estimate of *u*. If there is no knee, use the plotting position for the first data point.

Using a simple Weibull regression, calculate *b* and *h* for a Weibull line. An estimate for *h* and *H* is h = *H* = (*h*)2^(1/*b*) , if b = *B* = *b*.

If a Weibull fit yields a slope of 1.2 and a characteristic life of 100, the following values will result in an identical line to the simple Weibull fit for a competing risk model:

$\beta = B = 1.2$; $\eta = H = 178.2$

If a WeiBath model is used, use *b* and *h* calculated earlier. The line will not go through the data, but it will be sufficient for iterating. Before iterating, β, *B*, η, and *H* should be varied by 5-10% so they are not identical (b=1.2, β =1.1; B = 1.3, etc.) A relaxation technique may be used to improve an initial estimate.

MAXIMUM LIKELIHOOD ESTIMATE (MLE) – The likelihood of a particular set of parameters for a given observation of failures and suspensions is:

$$L = \prod_{j=1}^{n} f(t_j) \times \prod_{k=1}^{m} (1 - F(t_k)) \quad (12)$$

where the first product refers to failures and the second product refers to suspensions. With MLE, the parameters are chosen such that the likelihood function is maximized. This is typically solved by taking the logarithm and maximizing it:

$$\ln(L)=\sum_{j=1}^{n} \ln(f(t_j)) +\sum_{k=1}^{m} \ln(1-F(t_k)) \quad (13)$$

The instant probability of failure is the derivative of the cumulative probability of failure. The general equation for a bathtub distribution is:

$$f = u[f_i(1-F_c-F_w+F_cF_w)+F_i(-f_c-f_w+f_cF_w+F_cf_w)] + [f_c+f_w-f_cF_w-F_cf_w] \quad (14)$$

Taking the derivative of EQ(10), the instant probability of failure for a WeiBath distribution is:

$$f = (u)\mathbf{exp}[-(t/h)^b-(t/\eta)^\beta -(t/H)^B] \times[(b/h)(t/h)^{(b-1)} +(\beta/\eta)(t/\eta)^{(\beta-1)}+(B/H)(t/H)^{(B-1)}] +(1-u)\mathbf{exp}[-(t/\eta)^\beta -(t/H)^B] \times[(\beta/\eta)(t/\eta)^{(\beta-1)}+(B/H)(t/H)^{(B-1)}] \quad (15)$$

For Weibull analysis, EQ (10) and (15) are used in EQ (13). For other distributions, EQ(5) and EQ(14) are used.

MODEL SELECTION. – Fitting the wrong model (Competing risk versus competing risk mixture) to the data can result in erroneous conclusions. Just because a model has a better “fit” does not mean that it is a better choice. Similar data should be reviewed when making a selection.

ANALYSIS OF PARAMETERS – In a two or three parameter Weibull problem, meaning can be attached to the values of the final parameters. (If the Weibull slope is <1, it is a decreasing failure rate and probably represents an infant mortality type problem.)

This is not necessarily the case where failure modes are combined. Consider the equation $Y=A*X^{1.1}+B*X^{1.12}$. There are many choices of A and B that would give similar results if this model were used for regression.

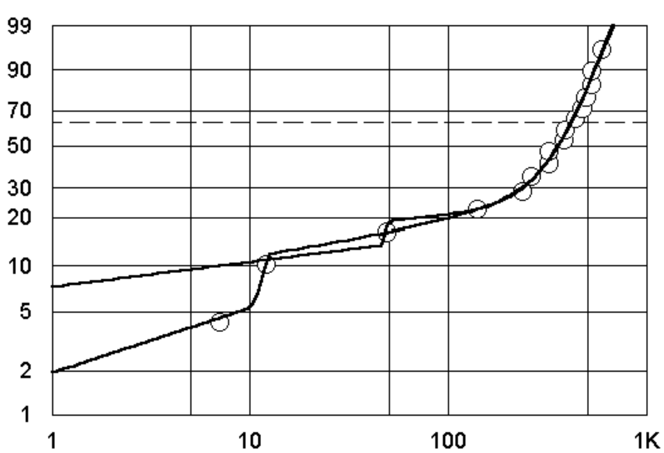

Figure 1. Two possible fit lines for a data set.

Consider the two sets of parameters and the predicted cumulative failure rate for the data shown in Table 1. The data points and fit lines, using EQ (10) are shown in Figure 1. These are selected points from Example 1. The fit parameters are drastically different, yet there is not a significant difference in the resulting values. “M.R” refers to the median ranks.

Table 1. Two possible fit lines, calculated y value, and plotting position (median rank)

	Set 1	Set 2
u	0.06600	0.06131
b	22.9506	136.035
h	11.8383	49.0658
β	0.4447	0.17152
η	6,959.08	3,438,339
B	3.95562	3.59877
H	466.347	453.999

Time, t	Set 1	Set 2	M.R.
12	0.1027	0.1027	0.1027
49	0.1637	0.1637	0.1637
140	0.2234	0.2246	0.2247
388	0.5632	0.5691	0.5918
592	0.9488	0.9443	0.9576

EXAMPLES

EXAMPLE 1. DISK DRIVES [3][6] – Dev Raheja reported on disk drive failures in his book. Sixteen failures occurred at t = 7, 12, 49, 140, 235, 260, 320, 320, 380, 388, 437, 472, 493, 524, 529, 592 hours.

The data were analyzed with a Weibull fit, competing risk (chance and wear out failures), competing risk mixture (infant and chance failures), and a WeiBath fit (infant, chance, and wear out). The results are listed in Tables 2-5, and Figures 2-5. Figures 2-5 are Weibull scale with cumulative percent on the vertical axis and hours on the horizontal axis.

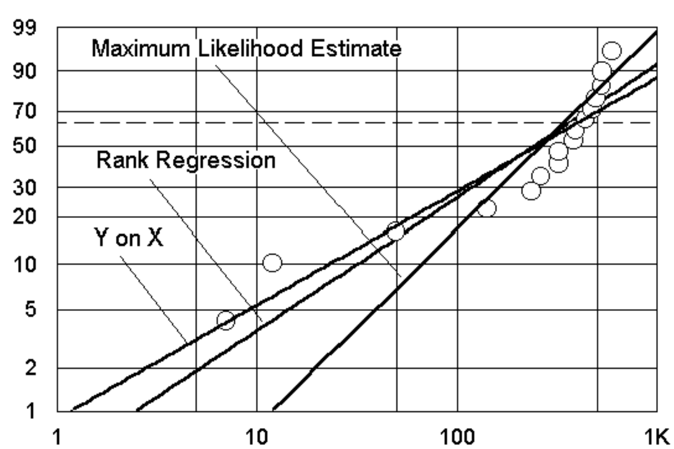

Figure 2. Weibull Fit, Disk Drives.

Table 2. Weibull Fit, Disk Drives

	Rank Reg	MLE	Y on X
β	0.9207	1.3679	0.7862
η	366.3	344.6	404.9
r²,X on Y	0.8539		0.8289
r²,Y on X	0.8289		0.8539

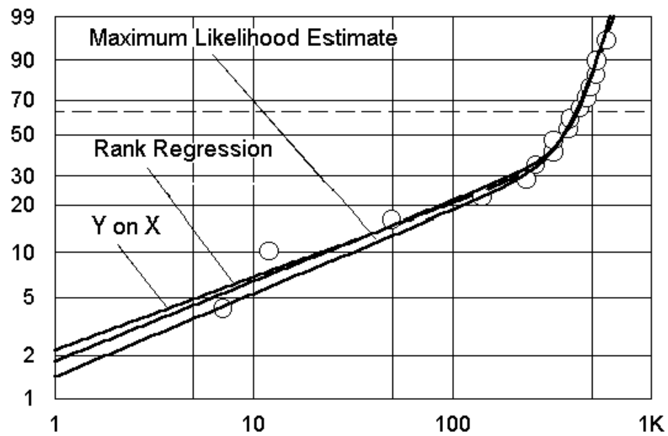

Figure 3. Competing Risk(Chance and wear out failures)

Table 3. Competing Risk(Chance and wear out failures)

	Rank Reg	MLE	Y on X
β	0.5587	0.5815	0.5108
η	1283.9	1500.4	1775.3
B	5.789	5.107	4.745
H	490.7	478.9	487.2
r²,X on Y	0.9703		0.9675
r²,Y on X	0.9820		0.9846

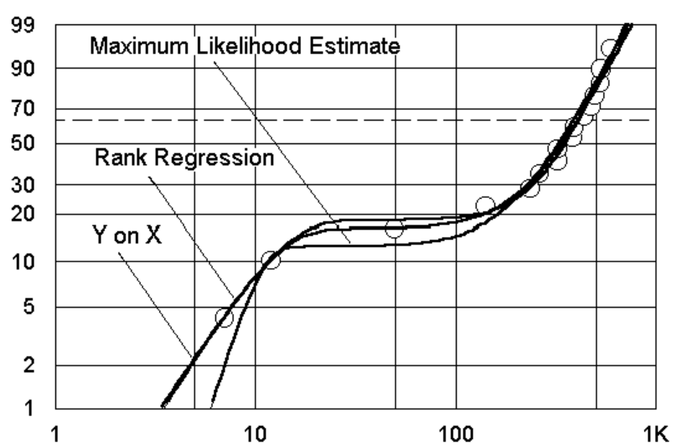

Figure 4. Competing Risk Mixture

Table 4. Competing Risk Mixture (Infant, chance failures)

	Rank Reg	MLE	Y on X
u	0.1612	0.1249	0.1824
b	2.226	4.451	2.117
h	11.93	10.47	13.12
β	2.661	2.608	3.159
η	434.9	410.3	441.7
r²,X on Y	0.9974		0.9665
r²,Y on X	0.9895		0.9928

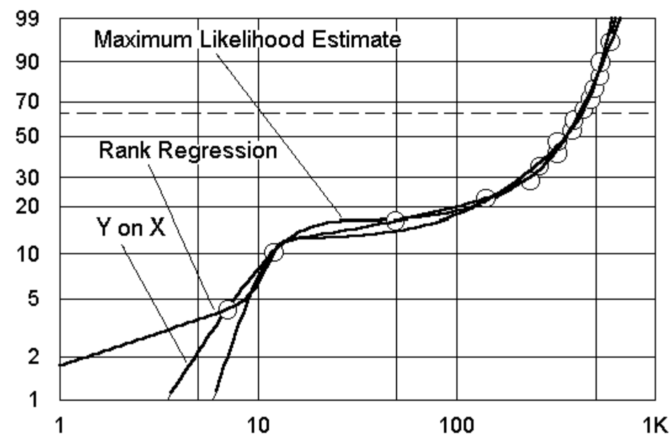

Figure 5. WeiBath fit

Table 5. WeiBath Fit (Infant, chance, wear out.)

	Rank Reg	MLE	Y on X
u	0.0708	0.1229	0.1611
b	9.485	4.451	2.226
h	11.64	10.47	11.93
β	0.463	1.640	2.661
η	6360.	530.5	504.7
B	3.966	8.529	7.508
H	466.8	530.7	554.4
r²,X on Y	0.9993		0.9952
r²,Y on X	0.9956		0.9965

For each fit, the analysis was done with Rank Regression, Maximum Likelihood Estimation, and Y on X Analysis. The coefficient of determination was also calculated for each set of parameters.

Note the differences in Figures 3 and 4. Figure 3 is based on competing risk, while Figure 4 is based on a competing risk mixture. The infant mortality “bump” can be seen between 7 and 50 hours. If the first failure had occurred at 1 hour, the graphs would be very similar. There are not enough early failures to justify using infant mortality in the model, but the graphs demonstrate the effects.

EXAMPLE 2. VALVE ASSEMBLIES – A new valve under development was tested on a fatigue test. Seventeen pieces were tested. Most valves were suspended, but two failure modes were observed: C-spring fracture and T-piece fracture. The valve was checked after each half schedule. The number of test schedules are listed below, with C indicating the C-spring, T indicating the T-piece, and S indicating suspension.

Data: 1.5C, 2C, 2.5T, 3.5x2C, 3.5T, 4x2T, 4.5T, 2.5x7S, 5S.

Since two failure modes are known, the failures are analyzed independently. (e.g., failures of the T-Piece are counted as suspensions for evaluation of C-spring life.) Figure 6 shows the plots of the two failure modes, the joint model using these two independent failure modes, and a lumped model that does not consider the failure mode, but considers only the time to failure.

Table 3 Shows the results of lumped data, individual data, and joint data, along with r^2 values for rank regression. While lumped data appears to give a good fit, comparison of the beta and eta values shows a significant difference in the T-piece slope parameters.

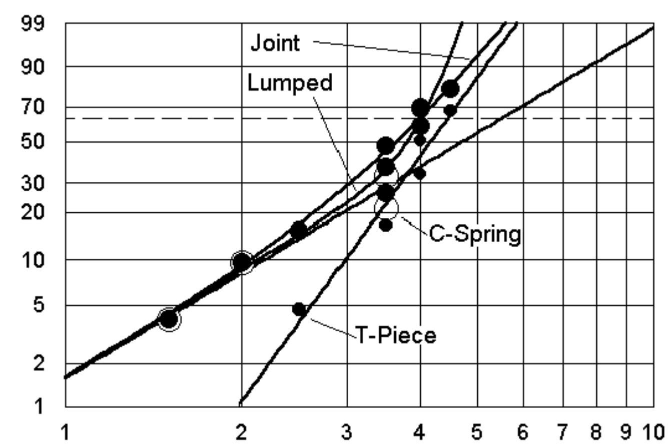

Figure 6. Valve Assembly, Two failure modes.

Table 6. Valve assembly

	Lumped	C-spring	T-Piece	Joint
β	2.4735	2.4222		2.4222
η	5.2914	5.5202		5.5202
B	12.6963		5.6075	5.6075
H	4.2579		4.4664	4.4664
r^2, X on Y	0.9891	0.9445	0.9521	0.9776
r^2, Y on X	0.9650	0.9412	0.9497	0.9470

EXAMPLE 3. LOCOMOTIVE POWER UNITS[3] – Each Southern Pacific diesel electric locomotive has 16 power units. Failure of a power unit results in a power loss and unscheduled replacement. Failures(66) and suspensions (238) for 304 units are listed by age in months and units failed. Negative age indicates suspension.

Data: 1x1, 3x2, 4x1, 9x2, 11x1, 17x2, 22x1, 23x3, 31x1 35x1, 36x1, 37x4, 39x3, 40x2, 44x2, 45x1, 46x3, 47x8, 48x2, 49x3, 50x6, 51x2, 52x14, -9x15, -12.5x16, -52.5x76, -53.5x131. (time in months)

The data were analyzed as a competing risk and as a competing risk mixture, using Maximum Likelihood. Values are shown in Table 4. Both approaches fit the data well.

There are at least three explanations. There may be an infant mortality sub-population. (Perhaps 8% of the power units have a different supplier for a key component.) There may be two types of failure modes that are affecting the units differently. A third explanation is that another measure, such as tonnage-miles, should be used instead of age. An engine gets one month of age whether it is parked, moves box-cars, or hauls coal over the mountains. The distribution may be a simple Weibull if the appropriate measure were used.

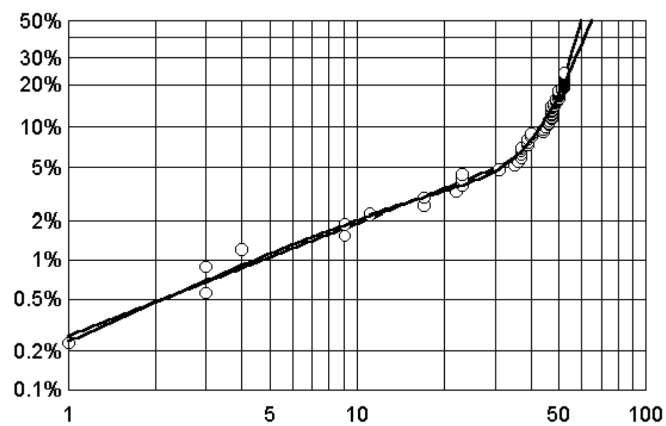

Figure 7. Locomotive Power Units

Table 7. Locomotive Power Units, 4 or 5 parameter fit

	Infant & Chance	Chance & Wear out
u	0.3709	
b	0.9096	
h	275.80	
β	8.7526	0.891
η	63.98	922.08
B		8.815
H		63.95

CONCLUSION

The bathtub curve can help model mixed failure modes where individual failure causes cannot be determined. The curve can be fit using rank regression or maximum likelihood. The resulting model may be used for predicting the failure rate at a given time.

When there is an apparent mixture of two failure modes, it can be modeled either as a competing risk mixture (infant mortality and chance), or it can be modeled as competing risk (chance and wear out).

For a given set of data, there may be several choices of parameters that fit equally well. Due to the nature of mixing failure modes, the resulting model parameters are not necessarily reliable indicators of the parameters of the true population.

While this paper has discussed failures in terms of a bathtub curve, the techniques are applicable to other combinations of failure modes.

ACKNOWLEDGMENTS

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ADDITIONAL SOURCES

Ybath™ was used to calculate the parameters in the examples. This program is available from:

- BathTub Software Inc. (ph 517-791-4405)
  e-mail: cdtarum@worldnet.att.net
  web site: http://home.att.net/~cdtarum/index.htm

Other software sources can be found by searching the Internet for "YBath", or "Weibull."

DEFINITIONS, ACRONYMS, ABBREVIATIONS

The following variables are used in the various equations:

β	Chance failures Weibull slope
η	Chance failures Weibull characteristic life
b	Infant mortality Weibull slope
B	Wear out failures Weibull slope
$f(t)$	Instantaneous failure rate
f_c	Instantaneous Failure rate for chance failures
f_i	Instantaneous Failure rate for infant mortality
f_r	Instantaneous Failure rate for remainder (1-u)
f_w	Instantaneous Failure rate for wear out
E	Sum of the squares of the error
$F(t)$	Cumulative Failure rate as function of time
F_c	Cumulative Failure rate for chance failures
F_i	Cumulative Failure rate for infant mortality
F_r	Cumulative Failure rate for remainder (1-u)
F_w	Cumulative Failure rate for wear out
h	Infant mortality Weibull characteristic life
H	Wear out failures Weibull characteristic life
m	Number of suspensions
n	Number of failures
R	Reliability at time t
R_k	Reliability of a system for failure mode k
r^2	Coefficient of determination
t	Time to failure or suspension
t_j	Time to failure
t_k	Time to suspension
t_o	Weibull minimum life parameter
u	Sub-population with infant mortality failures
Y	Individual observed values of Y
Y_c	Calculated values of Y
Y_{max}	Maximum Y value on a graph
Y_{min}	Minimum Y value on a graph

The following terms are used in this paper:

Weibath — The equation that results when a Weibull equation is used in the general bathtub reliability curve. See EQ (10)