APPENDIX D – DERIVATIONS FOR SECTION 4

THIS PAGE INTENTIONALLY LEFT BLANK.

APPENDIX D

This appendix utilizes the notation in Section 4.4.

Annex 1

We shall show the following:

(1) $E[I_i(t)] \;=\; 1-e^{-\lambda_i t}$

(2) $\mu(t;\underline{\lambda}) \;=\; \sum_{i=1}^{K}\left(1-e^{-\lambda_i t}\right)$

(3) $h(t;\underline{\lambda}) \;=\; \sum_{i=1}^{K}\lambda_i e^{-\lambda_i t}$

To show (1) observe that $I_i(t)$ is a random variable that only takes on the values zero and one. Thus

$$E[I_i(t)] \;=\; (0)\Pr(I_i(t)=0)+(1)\Pr(I_i(t)=1)$$

$$=\; \Pr(I_i(t)=1) \;=\; 1-e^{-\lambda_i t}$$

To show (2), let $M(t)$ denote the number of distinct B-modes that occur by t. Then

$$M(t) \;=\; \sum_{i=1}^{K} I_i(t)$$

Thus

$$\mu(t;\underline{\lambda}) \;=\; E(M(t)) \;=\; \sum_{i=1}^{K} E[I_i(t)] \;=\; \sum_{i=1}^{K}\left(1-e^{-\lambda_i t}\right)$$

Note (3) follows from (2) since

$$h(t;\underline{\lambda}) \;=\; \frac{d\,\mu(t;\underline{\lambda})}{d\,t} \;=\; \sum_{i=1}^{K}\lambda_i e^{-\lambda_i t}$$

Annex 2

Recall $\Lambda \sim \Gamma(\alpha,\beta)$. Let Ψ denote the moment generating function for Λ. Thus, by definition, $\Psi(x)=E\left(e^{x\Lambda}\right)$ for all real x for which the expectation with respect to Λ exists. One can show that Ψ is defined for $x < \frac{1}{\beta}$ and $\Psi(x) = (1 - \beta x)^{-(\alpha+1)}$ (see e.g. Mood and Graybill [9]). We shall utilize $\Psi(x)$ to express $\lambda_{B,K}$, $\mu(t)$, $h(t)$, $\rho(t)$, and $\theta(t)$ in terms of K and the gamma parameters α and β. We summarize our results below:

(1) $\lambda_{B,K} \quad = \quad K\beta(\alpha+1)$

(2) $\mu(t) \quad = \quad K\left[1-(1+\beta t)^{(\alpha+1)}\right]$

(3) $h(t) \quad = \quad \dfrac{K\beta(\alpha+1)}{(1+\beta t)^{\alpha+2}} \quad = \quad \dfrac{d\,\mu(t)}{d\,t}$

(4) $\rho(t) \quad = \quad \lambda_A + (1-\mu_d)K\beta(\alpha+1) + \dfrac{\mu_d\,K\beta(\alpha+1)}{(1+\beta t)^{\alpha+2}}$

(5) $\theta(t) \quad = \quad 1-(1+\beta t)^{-(\alpha+2)}$

To show (1), recall $\lambda_{B,K} = E\left(\sum_{i=1}^{K}\Lambda_i\right)$ where $\underline{\Lambda} = (\Lambda_1, \cdots, \Lambda_K)$ is a random sample from Λ. Thus

$$\lambda_{B,K} \quad = \quad K\,E(\Lambda) \quad = \quad K\frac{d\,\Psi(x)}{d\,x}\bigg|_{x=0}$$

$$= \quad K\beta(\alpha+1)$$

To demonstrate (2), note by Annex 1

$$\mu(t) \quad = \quad E[\mu(t;\underline{\Lambda})]$$

$$= \quad E\left[\sum_{i=1}^{K}\left(1-e^{-\Lambda_i t}\right)\right]$$

$$= \quad K - E\left\{\sum_{i=1}^{K} e^{-\Lambda_i t}\right\}$$

$$= \quad K - K\,E\left[e^{-\Lambda t}\right]$$

Thus

$$\mu(t) = K\{1 - E[e^{-\Lambda t}]\}$$

$$= K\{1 - \Psi(-t)\}$$

$$= K\{1 - (1 + \beta t)^{-(\alpha+1)}\}$$

To derive (3) we can utilize the expression for $h(t; \underline{\lambda})$ in Annex 1. Doing so we arrive at

$$h(t) = E[h(t; \underline{\Lambda})]$$

$$= E\left[\sum_{i=1}^{K} \Lambda_i e^{-\Lambda_i t}\right]$$

$$= K\, E[\Lambda e^{-\Lambda t}]$$

Note

$$E[\Lambda e^{-\Lambda t}] = \left.\frac{d\Psi}{dx}\right|_{x=-t}$$

$$= \frac{\beta(\alpha+1)}{(1+\beta t)^{\alpha+2}}$$

This yields

$$h(t) = \frac{K\,\beta(\alpha+1)}{(1+\beta t)^{\alpha+2}}$$

Note by (2) above,

$$\frac{d\,\mu(t)}{d\,t} = \frac{K\,\beta(\alpha+1)}{(1+\beta t)^{\alpha+2}}$$

Thus, as expected,

$$\frac{d\,\mu(t)}{d\,t} = E[h(t; \Lambda)] = h(t)$$

To obtain (4) we recall the expression in (16) of Section 4.4.3 for $\rho(t)$:

$$\rho(t) = \lambda_A + (1 - \mu_d)\lambda_{B.K} + \mu_d\, h(t)$$

Thus (4) directly follows from (1) and (3) above.

Finally, recall by (23) of Section 4.4.3 we have

$$\theta(t) \; = \; \frac{\lambda_{B,K} - h(t)}{\lambda_{B,K}}$$

By (1) and (3) above we note

$$h(t) \; = \; \frac{\lambda_{B,K}}{(1+\beta t)^{\alpha+2}}$$

Thus

$$\theta(t) \; = \; \frac{\lambda_{B,K} - \dfrac{\lambda_{B,K}}{(1+\beta t)^{\alpha+2}}}{\lambda_{B,K}}$$

$$= \; 1-(1+\beta t)^{-(\alpha+2)}$$

Annex 3

Maximum Likelihood Estimates for AMPM

To obtain maximum likelihood estimates (mle's) for our finite K and NHPP variants of the AMPM, assume m distinct B-modes first occur at test times $0 < t_1 \le t_2 \le \cdots \le t_m$ respectively over a test period of length T. Let n_A denote the number of A-mode failures that occur over test period T. We shall denote an estimate of a model parameter by placing the symbol "^" over the parameter. Thus, e.g., $\hat{\lambda}_A = n_A / T$ since λ_A is constant over test period T.

Let $\underline{t}$ be the vector of B-mode first occurrence times $(t_1, \cdots, t_m)$. Also, let $\langle K \rangle$ denote the set of positive integers less than or equal to K and let S_m denote the set of all subsets of $\langle K \rangle$ of size m. Then, conditioned on $\underline{\Lambda} = \underline{\lambda}$, the likelihood function for the test data $(m, \underline{t})$ is $L(m, \underline{t}; \underline{\lambda})$ where

$$L(m, \underline{t}; \underline{\lambda}) \; = m! \sum_{S \in S_m} \left[\prod_{i \in S} \lambda_i \, e^{-\lambda_i t_i} \prod_{i \in \langle K \rangle - S} e^{-\lambda_i T} \right] \qquad (1)$$

The summation in (1) is over all the mutually exclusive sets of exactly m distinct B-modes that can occur at first occurrence times $(t_1, \cdots, t_m)$.

Consider the corresponding likelihood random variable

$$L(m,\underline{t};\underline{\Lambda}) = m! \sum_{S\in S_m}\left[\prod_{i\in S}\Lambda_i e^{-\Lambda_i t_i} \prod_{i\in\langle K\rangle - S} e^{-\Lambda_i T}\right]$$

where $\underline{\Lambda} = (\Lambda_1, \ldots, \Lambda_K)$ is a random sample from $\Gamma(\alpha, \beta)$. Denote the expected value of L (m, $\underline{t}$; $\underline{\Lambda}$) with respect to $\underline{\Lambda}$ by L (m, $\underline{t}$). Since the Λ_i are independent and identically distributed for i = 1, …, K we have

$$L(m,\underline{t}) = m!\binom{K}{m}\left[\prod_{i=1}^{m} E\left(\Lambda e^{-\Lambda t_i}\right)\right]\left[E\left(e^{-\Lambda T}\right)\right]^{K-m} \tag{2}$$

where $\Lambda \sim \Gamma(\alpha, \beta)$. We wish to find the point (α, β) that maximizes L(m, $\underline{t}$). We shall denote these values of α and β by $\hat{\alpha}_K$ and $\hat{\beta}_K$, respectively.

By direct calculation of $E\left[\Lambda^p e^{u\Lambda}\right]$, recalling the form of density function f_Λ given in Section 4.4.2, we can show

$$E\left[\Lambda^p e^{u\Lambda}\right] = \frac{(\alpha+p)!\,\beta^p}{\alpha!(1-\beta u)^{\alpha+1+p}} \tag{3}$$

for $u < \beta^{-1}$ and $p > -(1+\alpha)$. From (2) and (3) we obtain

$$L(m,\underline{t}) = K(K-1)\cdots(K-m+1)\left[\frac{\beta^m(\alpha+1)^m}{(1+\beta T)^{(\alpha+1)(K-m)}}\right]\prod_{i=1}^{m}(1+\beta t_i)^{-(\alpha+2)} \tag{4}$$

Let $Z = \ln\{L(m,\underline{t})\}$. Then it follows that

$$\frac{\partial Z}{\partial\alpha} = \frac{m}{\alpha+1} - (K-m)\ln(1+\beta T) - \sum_{i=1}^{m}\ln(1+\beta t_i) \tag{5}$$

and

$$\frac{\partial Z}{\partial\beta} = \frac{m}{\beta} - \frac{(\alpha+1)(K-m)T}{1+\beta T} - (\alpha+2)\sum_{i=1}^{m}\frac{t_i}{1+\beta t_i} \tag{6}$$

Treating K as a positive real number we also obtain

$$\frac{\partial Z}{\partial K} = \sum_{i=0}^{m-1} \frac{1}{K-i} - (\alpha+1)\ln(1+\beta T) \qquad (7)$$

In Section 4.4 and this appendix we shall not use (7) since we are only interested in obtaining $\hat{\alpha}_K$, $\hat{\beta}_K$ in terms of K and the test data. We shall then hold the test data constant and let $K \to \infty$ to study the limiting behavior of our AMPM estimators. Let $\nu_K \underline{\underline{\Delta}} (\alpha, \beta, K)$ and $\hat{\nu}_K \underline{\underline{\Delta}} \left(\hat{\alpha}_K, \hat{\beta}_K, K\right)$. Then by (5) our maximum likelihood equation for α is

$$\left.\frac{\partial Z}{\partial \alpha}\right|_{\nu_K = \hat{\nu}_K} = 0 \Leftrightarrow \left(\hat{\alpha}_K + 1\right)^{-1} = m^{-1}\left[K \ln\left(1+\hat{\beta}_K T\right) - \sum_{i=1}^{m} \ln\left(\frac{1+\hat{\beta}_K T}{1+\hat{\beta}_K t_i}\right)\right] \qquad (8)$$

By (6) our maximum likelihood equation for β is

$$\left.\frac{\partial Z}{\partial \beta}\right|_{\nu_K = \hat{\nu}_K} = 0 \Leftrightarrow \hat{\alpha}_K + 1 = \frac{\dfrac{m}{\hat{\beta}_K} - \sum_{i=1}^{m} \dfrac{t_i}{1+\hat{\beta}_K t_i}}{\dfrac{(K-m)T}{1+\hat{\beta}_K T} + \sum_{i=1}^{m} \dfrac{t_i}{1+\hat{\beta}_K t_i}} \qquad (9)$$

Equating the expressions for $\left(\hat{\alpha}_K + 1\right)^{-1}$ obtained from (8) and (9) we arrive at a linear equation for K. Solving for K we obtain

$$K = \frac{\sum_{i=1}^{m} \ln \dfrac{1+\hat{\beta}_K T}{1+\hat{\beta}_K t_i} \sum_{i=1}^{m} \dfrac{1}{1+\hat{\beta}_K t_i} - \dfrac{m\hat{\beta}_K}{1+\hat{\beta}_K T} \sum_{i=1}^{m} \dfrac{T-t_i}{1+\hat{\beta}_K t_i}}{\ln\left(1+\hat{\beta}_K T\right) \sum_{i=1}^{m} \dfrac{1}{1+\hat{\beta}_K t_i} - \dfrac{m\hat{\beta}_K}{1+\hat{\beta}_K T} T} \qquad (10)$$

For a given K and data set $(m, \underline{t})$ generated over test period T we can solve (10) for $\hat{\beta}_K$. Then we can use either (8) or (9) to obtain $\hat{\alpha}_K$. Using $\left(\hat{\alpha}_K, \hat{\beta}_K, K\right)$ we can estimate all our finite K AMPM projection quantities where $\hat{\lambda}_A = n_A / T$ and μ_d is assessed as

$$\mu_d^{\bullet} \;=\; \frac{1}{m}\sum_{i \in obs} d_i^{\bullet} \tag{11}$$

In (11), $d_i^{\bullet}$ will often be based largely on engineering judgement. The value of $d_i^{\bullet}$ should reflect several considerations: (1) how certain we are that the problem has been correctly identified; (2) the nature of the fix, e.g., its complexity; (3) past FEF experience and (4) any germane testing (including assembly level testing).

In practice, we do not know the value of K. We could try to develop an mle for K based on (7) or by directly maximizing Z. We have found that a solution $\left(\hat{\alpha}, \hat{\beta}, \hat{K}\right)$ to the maximum likelihood equations (5), (6) and (7) can be a saddle point of $L(m,\underline{t})$. This can occur even for a large data set that appears to fit the model well. We present graphs in Section 4.4.6 for such a data set that clearly illustrate the difficulty in obtaining a reasonable estimate for K. Thus we prefer to take the point of view that we should not attempt to assess K. However, by conducting a standard failure modes and effects criticality analysis (FMECA), we can place a lower bound on K, say K_ℓ. Our experience with the AMPM is that if K is substantially higher than m, say, e.g., $K \geq 10m$, then our AMPM projection quantities will be insensitive to the value of K. We believe for a complex system or subsystem it will often be the case that $K_\ell \geq 10m$ or at least the unknown value of K will be $10m$ or higher. The factor of 10 may be larger than necessary. In practice, we suggest exercising the AMPM model with several plausible lower bound values for K and comparing the associated projections with those obtained in the limit as $K \rightarrow \infty$. This is illustrated for a data set in Section 4.4.6.

We shall now consider the behavior of our AMPM estimators as $K \rightarrow \infty$. To do so, let $\left\langle \hat{\beta}_K \right\rangle_{K>m}$ be a sequence satisfying (10) with limit $\hat{\beta}_\infty \in [0,\infty)$. We shall assume that such a sequence exists for our data set $(m,\underline{t})$ generated over $[0,T]$. Then by (10) we have

$$\ln\left(1+\hat{\beta}_\infty T\right)\sum_{i=1}^{m}\frac{1}{1+\hat{\beta}_\infty t_i} - \frac{m\hat{\beta}_\infty T}{1+\hat{\beta}_\infty T} \;=\; 0 \tag{12}$$

Recall by Annex 2, $\lambda_{B,K} = K\beta(\alpha+1)$, where we previously suppressed the subscript K. Thus we shall define $\hat{\lambda}_{B,K}$ by

$$\hat{\lambda}_{B,K} \;\underline{\Delta}\; K\hat{\beta}\left(\hat{\alpha}+1\right) \tag{13}$$

By (8) we obtain

$$\hat{\lambda}_{B,K} = K\hat{\beta}_K\left(\hat{\alpha}_K+1\right) = \frac{K m \hat{\beta}_K}{K\ln\left(1+\hat{\beta}_K T\right)-\sum_{i=1}^{m}\ln\frac{1+\hat{\beta}_K T}{1+\hat{\beta}_K t_i}} \tag{14}$$

Taking the limit in (14) as $K \to \infty$ we arrive at

$$\hat{\lambda}_{B,\infty} \underset{=}{\Delta} \lim_{K\to\infty}\hat{\lambda}_{B,K} = \frac{m\hat{\beta}_\infty}{\ln\left(1+\hat{\beta}_\infty T\right)} \tag{15}$$

provided $\hat{\beta}_\infty > 0$. If $\hat{\beta}_\infty = 0$, then we can show, by applying L'Hospital's rule, that the limit of the right hand side of (10) goes to a finite positive number as $K \to \infty$. This contradiction establishes that $\hat{\beta}_\infty > 0$. Since $K\hat{\beta}_K\left(\hat{\alpha}_K+1\right) \to \hat{\lambda}_{B,\infty} \in (0,\infty)$ as $K \to \infty$ and $\hat{\beta}_\infty \in (0,\infty)$, we obtain

$$\hat{\alpha}_\infty = \lim_{K\to\infty}\hat{\alpha}_K = -1 \tag{16}$$

We can now obtain our limiting AMPM estimates as $K \to \infty$. We first numerically solve (12) for $\hat{\beta}_\infty$ and then obtain $\hat{\lambda}_{B,\infty}$ from (15). From (16), the value of $\hat{\alpha}_\infty$ is -1. To go from the finite K AMPM estimate to the associated limiting estimate, we first consider $h_K(t)$ given by (3) in Annex 2, where we have suppressed the subscript K. Motivated by (3), we define

$$\hat{h}_K(t) \underset{=}{\Delta} \frac{K\hat{\beta}_K\left(\hat{\alpha}_K+1\right)}{\left(1+\hat{\beta}_K t\right)^{\hat{\alpha}_K+2}} \tag{17}$$

Then

$$\hat{h}_\infty(t) \underset{=}{\Delta} \lim_{K\to\infty}\hat{h}_K(t) = \frac{\hat{\lambda}_{B,\infty}}{1+\hat{\beta}_\infty t} \tag{18}$$

From (2) in Annex 2, we define

$$\hat{\mu}_K(t) \;\triangleq\; K\left[1-\left(1+\hat{\beta}_K t\right)^{-\left(\hat{\alpha}_K+1\right)}\right] \tag{19}$$

We can obtain $\hat{\mu}_\infty(t)$ more readily from (18) than from (19).

$$\hat{\mu}_\infty(t) \;\triangleq\; \lim_{K\to\infty}\hat{\mu}_K(t) \;=\; \lim_{K\to\infty}\int_0^t \hat{h}_K(x)\,dx \;=\; \int_0^t \lim_{K\to\infty}\hat{h}_K(x)\,dx$$

$$= \int_0^t \frac{\hat{\lambda}_{B,\infty}\,dx}{\left(1+\hat{\beta}_\infty x\right)} \;=\; \frac{\hat{\lambda}_{B,\infty}}{\hat{\beta}_\infty}\ln\left(1+\hat{\beta}_\infty t\right) \tag{20}$$

From (20), we can see that Equation (15) simply says

$$\hat{\mu}_\infty(T) \;=\; m \tag{21}$$

In accordance with (5) in Annex 2, we define

$$\hat{\theta}_K(t) \;\triangleq\; 1-\left(1+\hat{\beta}_K t\right)^{-\left(\hat{\alpha}_K+2\right)} \tag{22}$$

Then

$$\hat{\theta}_\infty(t) \;\triangleq\; \lim_{K\to\infty}\hat{\theta}_K \;=\; 1-\left(1+\hat{\beta}_\infty t\right)^{-1} \;=\; \frac{\hat{\beta}_\infty t}{1+\hat{\beta}_\infty t} \tag{23}$$

Finally, from (4) in Annex 2, we define

$$\hat{\rho}_K(t) \;\triangleq\; \hat{\lambda}_A+\left(1-\mu_d^{\bullet}\right)K\,\hat{\beta}_K\left(\hat{\alpha}_K+1\right)+\mu_d^{\bullet}\,\frac{K\,\hat{\beta}_K\left(\hat{\alpha}_K+1\right)}{\left(1+\hat{\beta}_K t\right)^{\hat{\alpha}_K+2}} \tag{24}$$

From (24) we have

$$\hat{\rho}_{\infty}(t) \triangleq \lim_{K \to \infty} \hat{\rho}_K(t) = \hat{\lambda}_A + \left(1 - \mu_d^{\bullet}\right)\hat{\lambda}_{B,\infty} + \mu_d^{\bullet}\left(\frac{\hat{\lambda}_{B,\infty}}{1 + \hat{\beta}_{\infty} t}\right) \tag{25}$$

Recall in Section 4.4.4 we showed our finite K AMPM converged to a NHPP in the sense that the process $\{X_K(t), 0 \le t < \infty\}$ converged to the NHPP $\{X_{\infty}(t), 0 \le t < \infty\}$ as $K \to \infty$. We also noted that $\{X_{\infty}(t), 0 \le t < \infty\}$ has the mean value function $\mu_{\infty}(t)$ given in (4.4.4). We could directly derive parameter estimators for this NHPP. By so doing, one can show that these estimators are identical to the limiting AMPM estimators.