APPENDIX C – DERIVATIONS FOR SECTION 2

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APPENDIX C
DERIVATIONS

Proposition 1.

$$f_{obs} \le c \iff TR \le \ell_\gamma(f_{obs})$$

Proof.

To prove this relation, we use the equation below which follows directly from the definition of a 100 γ percent lower confidence bound when f_{obs} failures occur in a demonstration test of length T_{Dem}:

$$\sum_{i=0}^{f_{obs}} e^{-T_{Dem}/\ell} \frac{(T_{Dem}/\ell)^i}{i!} = 1-\gamma$$

where

$$\ell \triangleq \ell_\gamma(f_{obs}).$$

Let g be the function of x > 0 defined by the left-hand side of the equation above with ℓ replaced by x. Note g is a strictly increasing function of x > 0 since g(x) is the probability of obtaining f_{obs} or fewer failures when the constant configuration under test has MTBF x.

I. First we shall show $f_{obs} \le c \Rightarrow TR \le \ell$.

Thus, let $f_{obs} \le c$. Suppose $\ell <$ TR. Then

$$g(\ell) < g(TR) = \sum_{i=0}^{f_{obs}} e^{-T_{Dem}/TR} \frac{(T_{Dem}/TR)^i}{i!}$$

$$\le \sum_{i=0}^{c} e^{-T_{Dem}/TR} \frac{(T_{Dem}/TR)^i}{i!}$$

$$\le 1-\gamma$$

which is a contradiction since $g(\ell) = 1-\gamma$. Thus, $TR \le \ell$.

II. Next we shall show $TR \le \ell \Rightarrow f_{obs} \le c$. Thus, let $TR \le \ell$. Suppose $f_{obs} > c$. Then

$$\sum_{i=0}^{f_{obs}} e^{-T_{Dem}/TR} \frac{(T_{Dem}/TR)^i}{i!} = g(TR) \le g(\ell) = 1-\gamma.$$

Since $f_{obs} > c$, this contradicts the definition of c (see Equation (5) in Section 2.1.2). Thus, $f_{obs} \le c$.

Proposition 2.

For each α<1, T>0, and M(T)>0, the corresponding distribution function of $L_\gamma(N,S)$ satisfies the inequality

$$\text{Prob}\left(L_\gamma(N,S) \le M(T) \right) \ge \gamma$$

Proof.

Let f_W denote the density function of W (defined by Equation (20) in Section 2.1.3) corresponding to α<1, T>0, M(T)>0. By inequality (21) in Section 2.1.3,

$$\text{Prob}\left(L_\gamma(N,S) \le M(T) \right)$$

$$= \int_0^\infty \left\{ \text{Prob}\left(L_\gamma(N,S;w) \le M(T) \right) \right\} f_w(w)dw$$

$$\ge \gamma \int_0^\infty f_w(w)\,dw$$

$$= \gamma$$

Proposition 3.

For each α<1, T>0, and M(T)>0,

$$\text{Prob}\left(L_\gamma(N,S) = x \right) = 0$$

for all real x.

Proof.

Let α<1, T>0, and M(T)>0. Clearly, $L_\gamma(N,S) \ge 0$. Thus, we need to consider $x \ge 0$.

Let $L_\gamma(n,S)$ denote $L_\gamma(N,S)$ conditioned on $N = n$. As shown in Appendix A of Reference 7,

$$\frac{\hat{M}_n(T)}{M(T)} \sim \left(\frac{\lambda T^\beta}{2n^2}\right)\chi^2_{2n}$$

where χ^2_ν is the chi-square random variable with ν degrees of freedom.

Thus,

$$\hat{M}_n(T) \sim \left(\frac{1}{\lambda\beta T^{\beta-1}}\right)\left(\frac{\lambda T^\beta}{2n^2}\right)\chi^2_{2n}$$

$$= \left(\frac{T}{2\beta n^2}\right)\chi^2_{2n}$$

Then, by (12) in Section 2.1.3,

$$L_\gamma(n,S) \sim \left(\frac{2n}{z_\gamma(n)}\right)^2\left(\frac{T}{2\beta n^2}\right)\chi^2_{2n}$$

i.e.,

$$L_\gamma(n,S) \sim \left(\frac{2T}{\beta}\right)\left(\frac{\chi^2_{2n}}{z^2_\gamma(n)}\right) \tag{34}$$

Thus,

$$\text{Prob}\,(L_\gamma(n,S) = x) = \text{Prob}\left(\chi^2_{2n} = \frac{\beta z^2_\gamma(n)x}{2T}\right) = 0$$

It then follows that,

$$\text{Prob}\,(L_\gamma\,(N,S) = x) =$$

$$[\text{Prob}(N=0)]^{-1}\sum_{n=1}^{\infty}[\text{Prob}(L_\gamma(n,S)=x)]\text{Prob}(N=n)$$

$$= 0, \text{ since Prob } (N=0) > 0.$$

Proposition 4.

Type II = $\text{Prob}(TR \le L_\gamma(N,S)) \le 1-\gamma$ for each $\alpha < 1$ and $T > 0$ where $M(T) = TR$.

Proof.

Let $\alpha < 1$ and $T > 0$ with $M(T) = TR$.

Then

$$\text{Prob}(TR \le L_\gamma(N,S)) =$$

$$\text{Prob}(L_\gamma(N,S) = TR) \quad + \quad \text{Prob}(TR < L_\gamma(N,S))$$

$$= \text{Prob}(TR < L_\gamma(N,S)), \text{ by Proposition 3,}$$

$$= 1 - \text{Prob}(L_\gamma(N,S) \le TR) \quad \le \quad 1-\gamma, \text{ by Proposition 2.}$$

Proposition 5.

For a growth curve with parameters (α, T, M(T)), the expected number of failures (E(N)) can be determined by

$$E(N) \quad = \quad \frac{T}{(1-\alpha)\,M(T)}$$

Proof.

The observed number of failures by test duration t, denoted by N(t), is a non-homogeneous Poisson process with $N(T) = N$ and intensity function

$$\rho(t) \quad = \quad \frac{1}{M(t)} \quad = \quad \lambda\beta t^{\beta-1}$$

This implies that N is Poisson distributed with expected value

$$E(N) \quad = \quad \int_0^T \rho(t)\,dt \quad = \quad \lambda T^{\beta}$$

By Equation (18) in Section 2.1.3,

$$E(N) \quad = \quad \frac{T^{\beta}}{(M(T))\beta T^{\beta-1}}$$

This yields

$$E(N) = \frac{T}{\beta M(T)} = \frac{T}{(1-\alpha)\ M(T)}.$$

Proposition 6.

For a growth curve with parameters (α, T, M(T)),

Prob (A; α, T, M(T)) =

$$(1-e^{-\mu})^{-1} \sum_{n=1}^{\infty} \left[\text{Prob}\left(\frac{\chi^2_{2n}}{z^2_\gamma\ (n)} \geq \frac{1}{2\mu d} \right) \right] e^{-\mu} \left(\frac{\mu^n}{n!} \right)$$

where $\mu \triangleq E(N)$ and $d \triangleq M(T)/TR$.

Proof.

From (23) in Section 2.1.3 and (34),

$$\text{Prob}(A; \alpha, T, M(T)) = \text{Prob}(L_\gamma(N,S) \geq TR)$$

$$= [1 - \text{Prob}\ (N = 0)]^{-1} \sum_{n=1}^{\infty} [\text{Prob}\ (L_\gamma\ (n,S) \geq TR)]\ \text{Prob}\ (N = n)$$

$$= [1 - \text{Prob}(N = 0)]^{-1} \sum_{n=1}^{\infty} \left[\text{Prob}\left(\left(\frac{2T}{\beta} \right) \left(\frac{\chi^2_{2n}}{z^2_\gamma(n)} \right) \geq TR \right) \right] \text{Prob}\ (N = n)$$

$$= [1 - \text{Prob}\ (N = 0)]^{-1} \sum_{n=1}^{\infty} \left[\text{Prob}\left(\frac{\chi^2_{2n}}{z^2_\gamma\ (x)} \geq \frac{\beta(TR)}{2T} \right) \right] \text{Prob}(N = n)$$

Letting $\mu \triangleq E(N)$ and $d \triangleq M(T)/TR$,

$$\text{Prob}(A; \alpha, T, M(T)) =$$

$$(1-e^{-\mu})^{-1} \sum_{n=1}^{\infty} \left[\text{Prob}\left(\frac{\chi^2_{2n}}{z^2_\gamma(n)} \geq (1/2) \left(\frac{\beta M(T)}{T} \right) \left(\frac{TR}{M(T)} \right) \right) \right] e^{-\mu} \left(\frac{\mu^n}{n!} \right)$$

$$= \quad (1-e^{-\mu})^{-1} \sum_{n=1}^{\infty} \left[\text{Prob}\left(\frac{\chi^2_{2n}}{z^2_\gamma\,(n)} \geq \frac{1}{2\mu d} \right) \right] e^{-\mu} \left(\frac{\mu^n}{n!} \right).$$