4. RELIABILITY GROWTH PROJECTION

4.1 Reliability Projection Concepts and Methodology. The reliability growth process applied to a complex system undergoing development involves surfacing failure modes, analyzing the modes, and implementing corrective actions (termed fixes) to the surfaced modes. In such a manner, the system configuration is matured with respect to reliability. The rate of improvement in reliability is determined by (1) the on-going rate at which new problem modes are being surfaced, (2) the effectiveness and timeliness of the fixes, and (3) the set of failure modes that are addressed by fixes.

At the end of a test phase, program management usually desires an assessment of the system’s reliability associated with the current configuration. Often, the amount of data generated from testing the current system configuration is severely limited. In such circumstances, if the failure data generated over a number of system configurations is consistent with a reliability growth model, we can pool the data over the tested configurations to estimate the parameters of the growth model. This in turn will yield a reliability tracking curve that gives estimates of the configuration reliabilities. The resulting assessment of the system’s current reliability is called a demonstrated estimate since it is based solely on test data.

If the current configuration is the result of applying a group of fixes to the previous configuration, there could be a statistical lack of fit in tracking reliability growth between the previous and current configurations. In such a situation it may not be valid to use a reliability growth tracking model to pool configuration data to assess the reliability of the current configuration. We always have the option of estimating the current configuration reliability based only on failure data generated for this configuration. However, such an estimate may be poor if little test time has been accumulated since the group of fixes was implemented. In this situation, program management may wish to use a reliability projection method. Such methods are typically based on assessments of the effectiveness of corrective actions and failure data generated from the current and previous configurations.

A second situation in which a reliability projection is often utilized is when a group of fixes are scheduled for implementation at the end of the current test phase, prior to commencing a follow-on test phase. Program management often desires a projection of the reliability that will be achieved by implementing the delayed fixes. This type of projection can be based solely on the current test phase failure data and engineering assessments of the effectiveness of the planned fixes. The Crow/AMSAA model in Section 4.3 or the AMSAA Maturity Projection Model (AMPM) discussed in Section 4.4 can be used to obtain such projections.

The current test phase could consist of several system configurations if not all the fixes to surfaced problem modes are delayed. In this instance we can still obtain a projection of the reliability with which the system will enter the follow-on test by using the AMPM.
Another situation in which a projection can be useful is in assessing the plausibility of meeting future reliability milestones, i.e., milestones beyond the commencement of the follow-on test. The AMPM can provide such projections based on failure data generated to date and fix effectiveness assessments for all implemented and planned fixes to surfaced problem modes.

In Section 4.2 we present several basic concepts used in connection with our reliability projection models. We also establish notation and present assumptions that are used throughout this section. Notation and assumptions directed toward a particular method are introduced in the corresponding section.

In Sections 4.3 and 4.4 we present two reliability projection models and associated statistical procedures. In Section 4.3 we discuss the Crow/AMSAA model. This model is used to estimate the system failure intensity at the beginning of a follow-on test phase based on information from the previous test phase. This information consists of problem mode first occurrence times, the number of failures associated with each problem mode, and the total number of failures due to modes that will not be addressed by fixes. Additionally, the projection uses engineering assessments of the planned corrective actions to problem modes surfaced during the test phase. The associated statistical estimation procedure assumes that all the corrective actions are implemented at the end of the current test phase but prior to commencing the follow-on test phase. This model addresses the continuous case, i.e., where test duration is measured in a continuous fashion such as in hours or miles.

In Section 4.4 we present another reliability projection model that addresses the continuous case. This model is called the AMSAA Maturity Projection Model – Continuous (AMPM-Continuous). The model can be applied to the situation where one wishes to utilize test data generated over one or more test phases to project the impact of fixes to surfaced problem failure modes. The model does not require that the fixes be all delayed to the end of the current test phase. It only assumes the fixes are implemented prior to the time at which a projection is desired. Also, projections may be made for milestones beyond the start of the next test phase. The section contains an example application of the AMPM.
4.2 Basic Concepts, Notation and Assumptions. Throughout this section we shall regard a potential failure mode as consisting of one or more potential failure sites with associated failure mechanisms. Fixes are often applied to failure modes surfaced through testing. As in Reference [1], we shall define a B-mode to be a failure mode we would apply a fix to if the mode were surfaced. All other failure modes will be referred to as A-modes. A surfaced mode might be regarded as an A-mode if (1) a fix is not economically justifiable, or (2) the underlying failure mechanisms associated with the mode are not sufficiently understood to attempt a fix. Thus the rate of failure due to the set of A-modes is constant as long as the failure modes are not reclassified.

For a surfaced B-mode, the rate of occurrence would hopefully diminish after implementing a fix to the mode. However, in general, we cannot expect the mode rate of occurrence to drop to zero. Fixes are seldom perfect; for example, our fix may not eliminate all the potential failure mechanisms associated with the B-mode. Thus, for each B-mode, say mode i, we associate a fix effectiveness factor (FEF), denoted by $d_i$. The FEF $d_i$ is the fraction by which the initial rate of occurrence of mode i is reduced due to the fix. The assessed values for the $d_i$ of surfaced B-modes are often based largely on engineering judgement. This is why the corresponding reliability assessment is termed a “projection” as opposed to a “demonstrated value” that is based solely on the test data.

List of Notation:

- $K$: Number of potential B-modes that reside in the system
- $\lambda_i$: Initial rate of occurrence of B-mode i ($i = 1, \ldots, K$)
- $\lambda_a$: Contribution of A-modes to system failure intensity
- $\lambda_b$: B-mode contribution to initial system failure intensity
- $T$: Total duration of conducted test. Typically measured in hours or miles.
- $N_{a,i}$: Number of A-mode failures that occur over $[0,T]$
- $N_{b,i}$: Number of B-mode failures that occur over $[0,T]$
- $m$: Number of distinct B-modes surfaced over $[0,T]$
- $M(t)$: Random variable of number of distinct B-modes surfaced by test duration t
- $\mu(t)$: The expected value of $M(t)$
- $t_i$: Time of first occurrence of B-mode i ($i = 1, \ldots, K$)
- $t$: Vector of B-mode first occurrence times ($t_1, \ldots, t_m$)
- $N_i$: Number of failures associated with B-mode i that occurs during test
- $d_i$: Fix effectiveness factor (FEF) for B-mode i. The factor $d_i$ is the fraction of $\lambda_i$ removed by the fix.
- $\text{obs}$: The index set associated with the $m$ B-modes that are surfaced during test
- $E$: Expectation operator
- $V$: Variance operator
\text{mle} \quad \text{Maximum likelihood estimator}
\wedge \quad \text{When placed over a parameter, it denotes an estimate}
\sim \quad \text{"Distributed as"}
\approx \quad \text{"Approximated by"}
\equiv \quad \text{"Approximately equal to"}

\textbf{Assumptions:}

1. At the start of test, there is a large unknown constant number, denoted by K, of potential B-modes that reside in the system (which could be a complex subsystem).

2. Failure modes (both types A and B) occur independently.

3. Each occurrence of a failure mode results in a system failure.

4. No new modes are introduced by attempted fixes.

Additional notation and assumptions germane to a particular model will be introduced in the section dealing with the model.

\subsection{4.3 Crow/AMSAA Reliability Projection Model.}

\subsubsection{4.3.1 Introduction.} In this section we shall consider the case where all fixes to surfaced B-modes are implemented at the end of the current test phase prior to commencing a follow-on test phase. Thus all fixes are delayed fixes. The current test phase will be referred to as Phase I and the follow-on test phase as Phase II.

The Crow/AMSAA reliability projection model and associated parameter estimation procedure was developed to assess the reliability impact of a group of delayed fixes. In particular, the model and estimation procedure allow assessment of what the system failure intensity will be at the start of Phase II after implementation of the delayed fixes. Denoting this failure intensity by \( \lambda(1) \), where \( \lambda \) denotes the duration of the test Phase I, the Crow/AMSAA assessment of \( \lambda(T) \) is based on: (1) the A and B mode failure data generated during Phase I test duration \( T \); and (2) assessments of the fix effectiveness factors (FEFs) for the B-modes surfaced during Phase I. Since the assessments of the FEFs are often largely based on engineering judgement, the resulting assessment, \( \hat{r}(T) \), of the system failure intensity after fix implementations is called a reliability projection as opposed to a demonstrated assessment (which would be based solely on test data).

The Crow/AMSAA projection model and estimation procedure was motivated by the desire to replace the widely used "adjustment procedure." The adjustment procedure assesses \( \lambda(T) \) based on reducing the number of failures \( N_i \) due to B-mode \( i \) during Phase I to \( \left(1 - d_i^*\right)N_i \), where \( d_i^* \) is the assessment of \( d_i \). Note \( \left(1 - d_i^*\right)N_i \) is an assessment of
the expected number of failures due to B-mode \( i \) that would occur in a follow-on test of the same duration as Phase I. The adjustment procedure assesses \( r(T) \) by \( \hat{r}_{\text{adj}}(T) \) where

\[
\hat{r}_{\text{adj}}(T) = \frac{N_i}{T} + \sum_{i=1}^{K} \left( 1 - d_i^* \right) \frac{N_i}{T}
\]

(1)

Crow [1] shows that even if the assessed FEFs are equal to the actual \( d_i \), the adjustment procedure systematically underestimates \( r(T) \). This bias, i.e.,

\[
B(T) = E[\hat{r}(T) - \hat{r}_{\text{adj}}(T)] > 0
\]

(2)

is calculated in [1] by considering the random set of B-modes surfaced during Phase I. In particular, the adjustment procedure is shown to be biased since it fails to take into account that, in general, not all the B-modes will be surfaced by the end of Phase I. Before discussing how the Crow/AMSAA methodology addresses this bias we shall list some additional notation and assumptions associated with the Crow/AMSAA model.

4.3.2 Crow/AMSAA Model Notation and Additional Assumptions.

List of Notation:

- \( D_i \): The conditional random variable for B-mode \( i \) (\( i = 1, \ldots, K \)) whose realization is the fix effectiveness factor \( d_i \) if mode \( i \) occurs during Test Phase I.
- \( \mu_d \): Expected value of \( D_i \).
- \( T \): Length of Test Phase I.
- \( r(T) \): System failure intensity at beginning of Test Phase II after implementation of delayed B mode fixes. Viewed as a random variable whose value is determined by the set of B-modes surfaced during Test Phase I and the associated fix effectiveness factors.
- \( \rho(T) \): Expected value of \( r(T) \) with respect to random set of B-modes surfaced in Test Phase I, conditioned on the fix effectiveness factor values. We write \( \rho(T) = E(r(T)) \).
- \( \hat{r}_{\text{adj}}(T) \): Adjustment procedure assessment of the value taken on by \( r(T) \).
- \( B(T) \): Bias incurred by assessing the value of \( r(T) \) by \( \hat{r}_{\text{adj}}(T) \). Thus, \( B(T) = E[\hat{r}(T) - \hat{r}_{\text{adj}}(T)] \).
- \( \rho_{GP} \): Growth potential system failure intensity.
- \( M_{GP} \): Growth potential system MTBF, i.e., \( M_{GP} = (\rho_{GP})^{-1} \).
$h(t)$ Expected rate of occurrence of new B-modes at test duration $t$. Note:

$$h(t) = \frac{d \mu(t)}{dt}$$

$h_c(t), r_c(t), \rho_c(t)$ Crow/AMSAA model approximations to $h(t), r(t), \rho(t)$ respectively

$M(T), M_c(T)$ Denote $(\rho(T))^{-1}$ and $(\rho_c(T))^{-1}$ respectively

Additional Assumptions for Crow/AMSAA:

1. The time to first occurrence is exponentially distributed for each failure mode.

2. No fixes to B-modes are implemented during Test Phase I. Fixes to all B-modes surfaced during Phase I are implemented prior to Phase II.

3. The fix effectiveness factors (FEFs) $d_i$ associated with the B-modes surfaced during Phase I are realized values of the random variables $D_i (i = 1, \ldots, K)$ where:
   (a) The $D_i$ are independent;
   (b) The $D_i$ have common mean value $\mu_d$; and
   (c) The $D_i$ are independent of $M(T)$.

4. The random process for the number of distinct B-modes that occur over test interval $[0, t]$, i.e. $M(t)$, is well approximated by a non-homogeneous Poisson process with mean value function $\mu_c(t) = \lambda t^\beta$ for some $\lambda, \beta > 0$.

4.3.3 Crow/AMSAA Model Equations and Estimation Procedure. The Crow/AMSAA model assesses the value of the system failure intensity, $r(T)$, after implementation of the Phase I delayed fixes. This assessment is taken to be an estimate of the expected value of $r(t)$, i.e., an estimate of $\rho(T) = E(r(T))$. In [1] (and in Section 4.4.3) it is shown that:

$$\rho(T) = \lambda_a + \sum_{i=1}^{K} (1 - d_i) \lambda_i + \sum_{i=1}^{K} d_i \lambda_i e^{-\lambda_a t}$$

(3)

The traditional adjustment procedure assessment for the value of $r(T)$ is actually an estimate of

$$\lambda_a + \sum_{i=1}^{K} (1 - d_i) \lambda_i$$

since as shown later in this subsection
\[
E(\hat{r}_{adj}(T)) = \lambda_i + \sum_{i=1}^{K} (1 - d_i) \lambda_i
\]  

(4)

where \( d_i^* \) is an assessment of \( d_i \). Thus, by (3) and (4), the adjustment procedure has the bias \( B(T) \) where

\[
B(T) = E[r(T) - \hat{r}_{adj}(T)]
\]

\[
= \rho(T) - E[\hat{r}_{adj}(T)]
\]

\[
= \sum_{i=1}^{K} (d_i^* - d_i) \lambda_i + \sum_{i=1}^{K} d_i \lambda_i e^{-\lambda T}
\]

It follows that for \( d_i^* = d_i \) (\( i = 1, \ldots, K \))

\[
B(T) = \sum_{i=1}^{K} d_i \lambda_i e^{-\lambda T}
\]  

(5)

This shows that even with perfect knowledge of the \( d_i \) (i.e., when \( d_i^* = d_i \)), the adjustment procedure provides a biased underestimate of the value of \( r(T) \). The Crow/AMSAA procedure attempts to reduce this bias by estimating \( B(T) \) given by (5).

To estimate \( B(T) \), the Crow/AMSAA Model uses an approximation to \( B(T) \). This approximation is obtained in two steps. The first step is to regard the \( d_i \) in (5) as realizations of random variables \( D_i \) (\( i = 1, \ldots, K \)) that satisfy assumption number 3 in the "Additional Assumptions for Crow/AMSAA." Then \( B(T) \) is approximated by the expected value (with respect to the \( D_i \)) of

\[
\sum_{i=1}^{K} D_i \lambda_i e^{-\lambda T}
\]

Thus the initial approximation arrived at for \( B(T) \) in (5) is

\[
B(T) = E\left( \sum_{i=1}^{K} D_i \lambda_i e^{-\lambda T} \right)
\]

\[
= \mu \sum_{i=1}^{K} \lambda_i e^{-\lambda T}
\]  

(6)
where $\mu_d = E(D_i) \ (i = 1, \ldots, K)$). The final step to obtain the Crow/AMSAA approximation of $B(T)$ is to replace the sum

$$\sum_{i=1}^{K} \lambda_i e^{-\lambda_i T}$$

in (6) by a two parameter function of $T$. The Crow/AMSAA Model replaces this sum by the power function

$$h_c(T) = \lambda \beta T^{\beta-1} \quad \text{for} \quad \lambda, \beta > 0 \quad (7)$$

The form in (7) is chosen based on the desire for a mathematically tractable estimation problem and an empirical observation. Based on an empirical study, Crow [1] states that the number of distinct B-modes surfaced over a test period $[0, t]$ can often be approximated by a power function of the form

$$\mu_c(t) = \lambda t^\beta \quad \text{for} \quad \lambda, \beta > 0 \quad (8)$$

In (8), Crow [1] interprets $\mu_c(t)$ as the expected number of distinct B-modes surfaced during the test interval $[0, t]$. More specifically, [1] assumes the number of distinct B-modes occurring over $[0, t]$ is governed by a non-homogeneous Poisson process with $\mu_c(t)$ as the mean value function. Thus

$$h_c(t) = \frac{d \mu_c(t)}{d t} = \lambda \beta t^{\beta-1} \quad (9)$$

represents the expected rate at which new B-modes are occurring at test time $t$.

In Annex 1 of Appendix D, under the previously stated assumptions, it is shown that the expected number of distinct B-modes surfaced over $[0, t]$ is given by

$$\mu(t) = \sum_{i=1}^{K} \left(1 - e^{-\lambda_i t}\right) \quad (10)$$

Thus the expected rate of occurrence of new B-modes at test time $t$ is

$$h(t) = \frac{d \mu(t)}{d t} = \sum_{i=1}^{K} \lambda_i e^{-\lambda_i t} \quad (11)$$

Equation (11) shows that the initial approximation to the bias $B(T)$, given in (6) can be expressed as
\[ B(T) \approx \mu_d h(T) \]  

(12)

By replacing \( h(T) \) in (12) by \( h_c(T) \) given in (9), we arrive at the final Crow/AMSAA Model approximation to \( B(T) \), namely

\[ B_c(T) = \mu_d h_c(T) \]

\[ = \mu_d \lambda \beta T^{\beta-1} \]  

(13)

 Returning to our expression in (3) for the expected value of the system failure intensity after incorporation of the Phase I delayed fixes, i.e., \( \rho(T) = E(r(T)) \), we can now write down the Crow/AMSAA Model approximation for \( \rho(T) \). This approximation, by (13), is given by:

\[ \rho_c(T) = \lambda_c + \sum_{i=1}^{K} (1-d_i) \lambda_i + B_c(T) \]

\[ = \lambda_c + \sum_{i=1}^{K} (1-d_i) \lambda_i + \mu_d \left( \lambda \beta T^{\beta-1} \right) \]  

(14)

We shall next consider the Crow/AMSAA procedure for estimating \( \rho_c(T) \). This estimate is taken as the assessment of the system failure intensity after incorporation of the delayed fixes.

Consider the first term in the expression for \( \rho_c(T) \) given in (14), i.e., \( \lambda_A \). Since the A-modes are not fixed, the A-mode failure rate \( \lambda_A \) is constant over \([0,T]\). Thus we simply estimate \( \lambda_A \) by

\[ \hat{\lambda}_A = \frac{N_A}{T} \]  

(15)

where \( N_A \) is the number of A-mode failures over \([0,T]\). Note

\[ E(\hat{\lambda}_A) = \frac{E(N_A)}{T} - \frac{\hat{\lambda}_A T}{T} = \hat{\lambda}_A \]  

(16)

Next consider estimation of the summation

\[ \sum_{i=1}^{K} (1-d_i) \lambda_i \]
in the expression for \( \rho_x(T) \). By the second assumption in the “Additional Assumptions for Crow/AMSAA,” all fixes are delayed until Test Phase I has been completed. This implies the failure rate for B-mode \( i (i = 1, \ldots, K) \) remains constant over \([0, T]\). Thus we simply estimate \( \hat{\lambda}_i \) by

\[
\hat{\lambda}_i = \frac{N_i}{T} \quad (i = 1, \ldots, K)
\]  

(17)

where \( N_i \) denotes the number of failures during \([0, T]\) attributable to B-mode \( i \). Note

\[
E(\hat{\lambda}_i) = \frac{E(N_i)}{T} = \frac{\hat{\lambda}_i T}{T} = \hat{\lambda}_i
\]

Equations (16) and (18) suggest we assess

\[
\lambda_A + \sum_{i=1}^{K} (1 - d_i) \hat{\lambda}_i
\]

by

\[
\hat{r}_{adj}(T) = \hat{\lambda}_A + \sum_{i=1}^{K} (1 - d_i) \hat{\lambda}_i
\]

\[
= \frac{N_A}{T} + \sum_{i=1}^{K} (1 - d_i) \left( \frac{N_i}{T} \right)
\]

(19)

Observe \( N_i = 0 \) if B-mode \( i \) does not occur during \([0, T]\). Thus

\[
\hat{r}_{adj}(T) = \frac{N_A}{T} + \sum_{i \in \text{obs}} (1 - d_i) \left( \frac{N_i}{T} \right)
\]

(20)

where \( \text{obs} = \{ i \mid \text{B-mode } i \text{ occurs during } [0, T] \} \). Note the adjustment procedure estimate has the form

\[
\hat{r}_{adj}(T) = \frac{N^*}{T}
\]

(21)

where

\[
N^* = N_A + \sum_{i \in \text{obs}} (1 - d_i) N_i
\]

(22)
is the "adjusted" number of failures.

For given fix effectiveness factor (FEF) assessments, \( d'_i \), note that

\[
E(\hat{r}_{ad}(T)) = T^{-1} \left\{ E(N_a) + \sum_{i=1}^{K} (1 - d'_i) E(N_i) \right\}
\]

\[
= T^{-1} \left\{ \lambda_a I + \sum_{i=1}^{K} (1 - d'_i) (\lambda_i I) \right\}
\]

\[
= \lambda_a + \sum_{i=1}^{K} (1 - d'_i) \lambda_i
\]

(23)

Thus, as stated earlier, we see that the adjustment procedure estimate only provides an assessment for a portion of the expected system failure intensity, namely

\[
\lambda_a = \sum_{i=1}^{K} (1 - d'_i) \lambda_i
\]

Returning to the fundamental equation for the Crow/AMSAA Model approximation to the expected system failure intensity, i.e. (14),

\[
\rho_a(T) = \lambda_a + \sum_{i=1}^{K} (1 - d'_i) \lambda_i + \mu_a \left( \lambda \beta T^{\beta - 1} \right)
\]

Let us next consider the assessment of the fix effectiveness factors \( d_i \). The assessment \( d'_i \) will often be based largely on engineering judgement. The value chosen for \( d'_i \) should reflect several considerations:

(1) How certain we are that the root cause for B-mode \( i \) has been correctly identified; (2) the nature of the fix, e.g., its complexity; (3) past FEF experience; and (4) any germane testing (including assembly level testing).

Note that (20) shows that we need only assess FEFs for those B-modes that occur during \([0,T]\) to make an assessment of

\[
\lambda_a + \sum_{i=1}^{K} (1 - d'_i) \lambda_i
\]

To assess the mean FEF, \( \mu_a = E(D') \), we utilize our assessments \( d'_i \) for \( i \in \text{obs} \).

Let \( m \) be the number of distinct B-modes surfaced over \([0,T]\). Then we assess \( \mu_a \) by
\[
\mu_{d} = \frac{1}{m} \sum_{i \in obs} d_{i}^{*}
\]  

(24)

To complete our assessment of the expected system failure intensity after incorporation of delayed fixes, we shall now address the assessment of

\[
h_{c}(T) = \lambda \beta T^{\beta-1}
\]

To develop a statistical estimation procedure for \(\lambda\) and \(\beta\), the Crow/AMSAA Model regards the number of distinct B-modes occurring in an interval \([0,t]\), denoted by \(M(t)\), as a random process. The model assumes that this random process can be well approximated, for large \(K\), by a non-homogeneous Poisson process with mean value function

\[
\mu_{c}(t) = E(M(t)) = \lambda t^{\beta}
\]

where \(\lambda, \beta, t > 0\). As noted earlier in (9)

\[
h_{c}(t) = \frac{d\mu_{c}(t)}{dt}
\]

The data required to estimate \(\lambda\) and \(\beta\) are (1) the number of distinct B-modes, \(m\), that occur during \([0,T]\) and (2) the B-mode first occurrence times \(0 < t_{1} \leq t_{2} \leq \cdots \leq t_{m} \leq T\). Crow [1] states that the maximum likelihood estimates of \(\lambda\) and \(\beta\), denoted by \(\hat{\lambda}\) and \(\hat{\beta}\) respectively, satisfy the following equations:

\[
\hat{\lambda} T^{\hat{\beta}} = m
\]  

(25)

\[
\hat{\beta} = \frac{m}{\sum_{i=1}^{m} \ln \left( \frac{T}{t_{i}} \right)}
\]  

(26)

Note (25) merely says that the estimated number of distinct B-modes that occur during \([0,T]\) should equal the observed number of distinct B-modes over this period. Solving (25) for \(\hat{\lambda}\) we can write our estimate for \(h_{c}(T)\) in terms of \(m\) and \(\hat{\beta}\) as follows:

\[
h_{c}(T) = \hat{\lambda} \hat{\beta} T^{\hat{\beta}-1} = \left( \frac{m}{T^{\hat{\beta}}} \right) \hat{\beta} T^{\hat{\beta}-1}
\]

\[
= \frac{m \hat{\beta}}{T}
\]  

(27)
Crow [1] notes that conditioned on the observed number of distinct B-modes \( m \), i.e. \( M(T) = m \), the estimator

\[
\bar{\beta}_m = \left( \frac{m-1}{m} \right) \hat{\beta} \quad m > 2
\]  

(28)

is an unbiased estimator of \( \beta \), i.e.,

\[
E(\bar{\beta}_m) = \beta
\]  

(29)

Thus we shall also consider estimating \( h_c(T) = \lambda \beta T^{\beta-1} \) by using \( \bar{\beta}_m \). This leads to the estimate

\[
\bar{h}_c(T) = \frac{m \bar{\beta}_m}{t}
\]  

(30)

Finally, to complete our assessment of the system failure intensity, we need to assess the Crow/AMSAA Model expected system failure intensity \( \rho_c(T) \). Recall, by (14)

\[
\rho_c(T) = \lambda + \sum_{i=1}^{K} (1-d_i) \lambda_i + \mu_d h_c(T)
\]  

(31)

Piecing together our assessments for the individual terms in (31) we arrive at the following assessment for \( \rho_c(T) \) based on \( \hat{\beta} \):

\[
\hat{\rho}_c(T) = \frac{N_i}{T} + \sum_{i=1}^{K} (1-d_i) \left( \frac{N_i}{T} \right) + \left( \frac{1}{m} \sum_{i \in \text{obs}} d_i^* \right) \left( \frac{m \hat{\beta}}{T} \right)
\]

\[
= \frac{1}{T} \left\{ N_i + \sum_{i=1}^{K} (1-d_i^*) N_i + \hat{\beta} \sum_{i \in \text{obs}} d_i^* \right\}
\]

Since \( N_i = 0 \) for \( i \not\in \text{obs} \), we finally obtain

\[
\hat{\rho}_c(T) = \frac{1}{T} \left\{ N_i + \sum_{i \in \text{obs}} (1-d_i^*) N_i + \hat{\beta} \sum_{i \in \text{obs}} d_i^* \right\}
\]  

(32)

Likewise, we arrive at the following alternate assessment for \( \rho_c(T) \) based on \( \bar{\beta}_m \) (provided \( m \geq 2 \)):
\[
\hat{\rho}_c(T) = \frac{1}{T} \left\{ N_4 + \sum_{i \leq 0.08} (1 - d_i) N_i + \bar{\beta}_m \sum_{i \leq 0.08} d_i \right\}
\] (33)

Note both estimates of \( \rho_c(T) \) are of the form

\[
\text{Estimate } \rho_c(T) = \frac{1}{T} \left\{ N^* + \text{(estimate } \beta) \sum_{i \leq 0.08} d_i \right\}
\] (34)

where \( N^* \) is the “adjusted” number of failures over \([0,T]\). Recall the historically used adjustment procedure assessment for the system failure intensity, after incorporation of delayed fixes, is given by

\[
\hat{\hat{\alpha}}_{\text{adj}}(T) = \frac{N^*}{T}
\]

Also recall

\[
\bar{\beta}_m = \left( \frac{m-1}{m} \right) \hat{\beta} < \hat{\beta}
\]

Thus we see by (32) and (33)

\[
\hat{\hat{\alpha}}_{\text{adj}}(T) < \hat{\rho}_c(T) < \dot{\rho}_c(T)
\] (35)

Also of interest is an assessment of the reciprocal of \( \rho_c(T) \), i.e.

\[
M_c(T) = \{\rho_c(T)\}^{-1}
\]

The assessment for the system mean time between failures after incorporation of the delayed fixes, denoted by \( M(T) \), is taken to be the Crow/AMSAA Model assessment of \( M_c(T) \). The assessments of \( M_c(T) \) based on \( \dot{\rho}_c(T) \) and \( \hat{\rho}_c(T) \) are denoted by \( \dot{M}_c(T) \) and \( \hat{M}_c(T) \) respectively. Thus

\[
\dot{M}_c(T) = \{\dot{\rho}_c(T)\}^{-1}
\] (36)

and

\[
\hat{M}_c(T) = \{\hat{\rho}_c(T)\}^{-1}
\] (37)

By (35) we have
\( \dot{M}_c(T) < \ddot{M}_c(T) < \left( \dot{\hat{\rho}}_{adj}(T) \right)^{-1} \)

In Section 4.3.5 we shall argue that \( \ddot{\rho}_c(T) \) generally provides a more accurate assessment of \( \rho_c(T) \) than does \( \dot{\hat{\rho}}_c(T) \). However, somewhat surprisingly at first thought, in Section 4.3.5 we identify conditions under which \( \dot{M}_c(T) \) generally provides a more accurate assessment of \( M_c(T) \) than does \( \ddot{M}_c(T) \).

4.3.4 Reliability Growth Potential. Consider the expression in (3) for \( \rho(T) \), the expected system failure intensity after incorporation of the delayed fixes. If we let \( T \to \infty \) and denote the resulting limit of \( \rho(T) \) by \( \rho_{GP} \) we obtain

\[
\rho_{GP} = \lim_{T \to \infty} \rho(T) = \lambda_i + \sum_{i=1}^{K} (1-d_i) \lambda_i
\]

The expression \( \rho_{GP} \) is called the growth potential failure intensity. Its reciprocal is referred to as the growth potential MTBF. The growth potential MTBF represents a theoretical upper limit on the system MTBF. This limit corresponds to the MTBF that would result if all B-modes were surfaced and corrected with specified fix effectiveness factors. Note \( \rho_{GP} \) is estimated by

\[
\hat{\rho}_{GP} = \frac{1}{T} \left( N_A + \sum_{i \text{obs}} (1-d_i^*) N_i \right)
\]

If the reciprocal \( (\hat{\rho}_{GP})^{-1} \) lies below the goal MTBF then this may indicate that achieving the goal is high risk.

4.3.5 Use of the Maximum Likelihood Estimator versus the Unbiased Estimator for \( \beta \). Recall that the estimator

\[
\overline{\beta}_m = \left( \frac{m-1}{m} \right) \hat{\beta}
\]

conditioned on \( M(T) = m \), with \( m \geq 2 \), is unbiased for \( \beta \), i.e.

\[
E(\overline{\beta}_m) = \beta
\]

Furthermore the variances of \( \overline{\beta}_m \) and \( \hat{\beta} \), denoted by \( V(\overline{\beta}_m) \) and \( V(\hat{\beta}) \) respectively, satisfy the following:
\[
V(\bar{\beta}_m) = V\left(\left(\frac{m-1}{m}\right)\hat{\beta}\right) \\
= \left(\frac{m-1}{m}\right)^2 V(\hat{\beta}) < V(\hat{\beta})
\] (41)

for \( m \geq 2 \). Equation (41) together with the unbiased property of \( \bar{\beta}_m \), suggest that \( \bar{\beta}_m \) provides a more accurate assessment of \( \beta \) than does \( \hat{\beta} \).

Next consider the assessments of \( h_c(T) \) based on \( \hat{\beta} \) and \( \bar{\beta}_m \). Recall the Crow/AMSAA Model assumes that \( M(t), t>0 \), is a non-homogeneous Poisson process with mean value function

\[
\mu_c(t) = E(M(t)) = \lambda t^\beta, \quad \lambda, \beta > 0
\]

Thus, in particular, \( M(T) \) is Poisson distributed with mean

\[
E(M(T)) = \lambda T^\beta
\]

Using this fact, it can be shown that \( \bar{h}_c(T) \) is an approximately unbiased estimator of \( h_c(T) \) under most conditions of practical interest, where it is understood that \( \bar{h}_c(T) \) denotes a conditional estimator, conditioned on \( M(T) > 0 \). To be more explicit, \( \bar{h}_c(T) \) when viewed as an estimator (as opposed to an estimated value), is a random variable which is a function of \( M(T) \) and the random vector of B-mode first occurrence times \( (T_1, \ldots, T_{M(T)}) \). When \( M(T) = m \) and \( (T_1, \ldots, T_{M(T)}) = (t_1, \ldots, t_m) \), the estimator \( \bar{h}_c(T) \) takes on the value \( \frac{m \bar{\beta}_m}{T} \)

where

\[
\bar{\beta}_m = \left(\frac{m-1}{m}\right)\hat{\beta} = \left(\frac{m-1}{m}\right) \left( \frac{m}{\sum_{i=1}^m \ln \left(\frac{T}{t_i}\right)} \right) = \frac{m-1}{\sum_{i=1}^m \ln \left(\frac{T}{t_i}\right)}
\]

The estimator \( \bar{h}_c(T) \) can be shown to satisfy the following:

\[
E(\bar{h}_c(T)) \equiv h_c(T)
\] (42)
provided $\Pr(M(T) = 0) \equiv 0$, where $\Pr$ denotes the probability function for $M(1)$.

Consider the variances of the estimators $\overline{h}_c(T)$ and $\hat{h}_c(T)$ conditioned on $M(T) = m$. For $m \geq 2$,

$$V(\overline{h}_c(T) | M(T) = m) = V\left(\frac{m \overline{\beta}_m}{T}\right)$$

$$= \left(\frac{m}{T}\right)^2 V(\overline{\beta}_m) = \left(\frac{m}{T}\right)^2 V\left(\left(\frac{m-1}{m}\right)\hat{\beta}\right)$$

$$= \left(\frac{m}{T}\right)^2 \left(\frac{m-1}{m}\right)^2 V(\hat{\beta}) = \left(\frac{m-1}{T}\right)^2 V(\hat{\beta})$$

$$< \left(\frac{m}{T}\right)^2 V(\hat{\beta}) = V\left(\frac{m \hat{\beta}}{T}\right)$$

$$= V(\hat{h}_c(T) | M(T) = m)$$

(43)

Now consider the variances of $\overline{h}_c(T)$ and $\hat{h}_c(T)$ conditioned on $M(T) \geq 2$. Since (43) holds for each $m \geq 2$, we have

$$V(\overline{h}_c(T) | M(T) \geq 2) < V(\hat{h}_c(T) | M(T) \geq 2)$$

(44)

Equations (42) and (44) suggest that the estimator $\overline{h}_c(T)$ provides a more accurate estimate of $h_c(T)$ than does the estimator $\hat{h}_c(T)$ when two or more distinct B-modes occur during [0, T].

We now investigate the bias of the estimators $\overline{\rho}_c(T)$ and $\overline{\rho}_c(T)$. To do so, let

$$\overline{\rho}_c(T) = \frac{1}{T} \left\{ N_A + \sum_{i \in \text{oobs}} (1 - d_i) N_i + \overline{\beta} \sum_{i \in \text{oobs}} d_i \right\}$$

where $\overline{\beta} \in \{\hat{\beta}, \overline{\beta}_m\}$. Also let $\overline{h}_c(T) = \frac{m \overline{\beta}}{T}$. By (27) and (30) we have
\[ \tilde{\rho}_c(T) = \left\{ \frac{N_A}{T} + \sum_{i=1}^{K} (1 - d_i) \frac{N_i}{T} + \left( \frac{1}{m} \sum_{i=obs} d_i \right) \left( \frac{m \beta}{T} \right) \right\} \]

\[ = \frac{N_A}{T} + \sum_{i=1}^{K} (1 - d_i) \frac{N_i}{T} + \mu_d \tilde{h}_c(T) \]

Thus the expected value of \( \tilde{\rho}_c(T) \) is

\[ E(\tilde{\rho}_c(T)) = \lambda_c + \sum_{i=1}^{K} (1 - d_i) \lambda_i + E(\mu_d \tilde{h}_c(T)) \]  \hspace{1cm} (45)

Recall by (31),

\[ \rho_c(T) = \lambda_c + \sum_{i=1}^{K} (1 - d_i) \lambda_i + \mu_d h_c(T) \]

Thus by Equation (45), we have

\[ E(\tilde{\rho}_c(T)) - \rho_c(T) = \sum_{i=1}^{K} (d_i - d_i^*) \lambda_i + E(\{ \mu_d^* - \mu_d \} \tilde{h}_c(T)) + \mu_d \{ E(\tilde{h}_c(T)) - h_c(T) \} \]  \hspace{1cm} (46)

By (46) we see that even if our assessments of \( \mu_d \) and the \( d_i \) are perfect, the estimator \( \tilde{\rho}_c(T) \) will have a residual bias of

\[ \mu_d \{ E(\tilde{h}_c(T)) - h_c(T) \} \]

To reduce this residual bias as much as possible, we wish to make the bias

\[ E(\tilde{h}_c(T)) - h_c(T) \]

as small as possible. Since \( \tilde{h}_c(T) \) is almost an unbiased estimator for \( h_c(T) \), this suggests we use

\[ \tilde{\rho}_c(T) = \frac{1}{T} \left\{ N, + \sum_{i=obs} (1 - d_i^*) N_i + \tilde{\beta} \sum_{i=obs} d_i^* \right\} \]

to assess \( \rho_c(T) \).
Next, we discuss the assessment of \( M_c(T) = \{ \rho_c(T) \}^{-1} \). To do so, let

\[
\tilde{M}_c(T) = \{ \tilde{\rho}_c(T) \}^{-1}.
\]

Thus

\[
\tilde{M}_c(T) = \left[ \frac{1}{T} \left( N_A + \sum_{i \in \text{obs}} (1 - d_i) N_i + \beta \sum_{i \in \text{obs}} d_i \right) \right]^{-1}
\]

\[
= \left[ \frac{N_A}{T} + \sum_{i \in \text{obs}} (1 - d_i) \frac{N_i}{T} + \left( \frac{1}{m} \sum_{i \in \text{obs}} d_i^* \right) \tilde{h}_c(T) \right]^{-1} \tag{47}
\]

Also

\[
M_c(T) = \left[ \lambda_A + \sum_{i=1}^K (1 - d_i) \lambda_i + \mu \tilde{h}_c(T) \right]^{-1} \tag{48}
\]

We have shown that to minimize the bias

\[
E \{ \tilde{\rho}_c(T) \} - \rho_c(T)
\]

we should use \( \tilde{\rho}_c(T) \) instead of \( \rho_c(T) \) to estimate \( \rho_c(T) \). However, we wish to demonstrate that one should not infer from this that \( M_c(T) \) must have a smaller bias than \( \tilde{M}_c(T) \) as an estimator of \( M_c(T) \). To demonstrate this we shall consider a simple case, the instance when the bias of \( \tilde{M}_c(T) \) is approximately equal to the bias of \( \{ \tilde{h}_c(T) \}^{-1} \).

Thus in the following assume

\[
E \left[ \tilde{M}_c(T) \mid M(T) \geq 2 \right] - M_c(T)
\]

\[
\equiv E \left[ \{ \tilde{h}_c(T) \}^{-1} \mid M(T) \geq 2 \right] - \{ h_c(T) \}^{-1} \tag{49}
\]

One instance where (49) would be expected to hold is when \( \lambda_A \equiv 0 \) and \( d_i \equiv 1 \) for \( i = 1, \ldots, K \). For such conditions, we have by (48) that \( M_c(T) \equiv \{ h_c(T) \}^{-1} \). Also, in such a case, it is reasonable that \( d_i^* \equiv 1 \) for \( i \in \text{obs} \) and, with high probability, \( \frac{N_A}{T} \equiv 0 \). By (47) we see that such conditions would imply
\[ E \left[ \tilde{M}_c (T) | M(T) \geq 2 \right] \]

\[ \approx E \left[ \tilde{h}_c (T)^{-1} | M(T) \geq 2 \right] \]

The above expectations and all subsequent expectations in this section are with respect to all the random quantities for given \( d^i \), conditioned on \( M(T) \geq 2 \). These random quantities are the number of A-mode failures and the number of distinct B-modes experienced over \([0, T]\), and the random vector of B-mode first occurrence times \((T_1, \ldots, T_{M(T)})\).

Now consider the expected values of \( \{ \tilde{h}_c (T) \}^{-1} \) and \( \{ \tilde{h}_c (T) \}^{-1} \) conditioned on \( M(T) \geq 2 \). From the fact that the number of distinct B-modes occurring over \([0,T]\) is Poisson with mean \( \lambda T^b \) it can be shown

\[ \{ h_c (T) \}^{-1} < E \left[ \tilde{h}_c (T)^{-1} | M(T) \geq 2 \right] < E \left[ \tilde{h}_c (T)^{-1} | M(T) \geq 2 \right] \]

(50)

for \( \mu(T) \geq 3.2 \). Thus, when (49) holds, Equation (50) implies

\[ E \left[ \tilde{M}_c (T) | M(T) \geq 2 \right] - M_c (T) \]

\[ \approx E \left[ \tilde{h}_c (T)^{-1} | M(T) \geq 2 \right] - \{ h_c (T) \}^{-1} \]

\[ > E \left[ \tilde{h}_c (T)^{-1} | M(T) \geq 2 \right] - \{ h_c (T) \}^{-1} \]

\[ \approx E \left[ \tilde{M}_c (T) | M(T) \geq 2 \right] - M_c (T) \]

(51)

and

\[ E \left[ \tilde{M}_c (T) | M(T) \geq 2 \right] - M_c (T) \]

\[ \approx E \left[ \tilde{h}_c (T)^{-1} | M(T) \geq 2 \right] - \{ h_c (T) \}^{-1} > 0 \]

(52)
Equations (51) and (52) show that for the case considered we should anticipate that \( \hat{M}_c(T) \) and \( \bar{M}_c(T) \) will have positive biases with the bias of \( \bar{M}_c(T) \) larger than that of \( M_c(T) \). It has not been established whether this holds more generally. If there is concern that \( \hat{M}_c(T) \) will have a positive bias and that the bias of \( \bar{M}_c(T) \) will exceed that of \( M_c(T) \), then one may wish to assess \( M_c(T) \) by the more conservative estimator \( \hat{M}_c(T) \) (recall \( \hat{M}_c(T) < \bar{M}_c(T) \) for \( M(T) \geq 2 \)).

4.3.6 Example. The following example is taken from [1] and illustrates application of the Crow/AMSAA model.

Data were generated by a computer simulation with \( \lambda_A = 0.02 \), \( \lambda_g = 0.1 \), \( K = 100 \) and the \( d_i \)'s distributed according to a beta distribution with mean 0.7. The simulation portrayed a system tested for \( T = 400 \) hours. The simulation generated \( N = 42 \) failures with \( N_A = 10 \) and \( N_g = 32 \). The thirty-two B-mode failures were due to \( M=16 \) distinct B-modes. The B-modes are labeled by the index \( i \) where the first occurrence time for mode \( i \) is \( t_i \) and \( 0 < t_1 < t_2 < \cdots < t_{16} < T = 400 \).

Table 1 lists, for each B-mode \( i \), the time of first occurrence followed by the times of subsequent occurrences (if any). Column 3 of the table lists \( N_i \), the total number of occurrences of B-mode \( i \) during the test period. Column 4 contains the assessed fix effectiveness factors for each of the observed B-modes. Column 5 has the assessed expected number of type \( i \) B-modes that would occur in \( T=400 \) hours after implementation of the fix. Finally, the last column contains the base e logarithms of the B-mode first occurrence times. These are used to calculate \( \hat{\beta} \).
Table 1. Projection Example Data.

<table>
<thead>
<tr>
<th>B mode</th>
<th>Failure Times (hrs)</th>
<th>(N_i)</th>
<th>(d_i^*)</th>
<th>((1 - d_i^*)N_i)</th>
<th>(\ln t_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15.04, 254.99</td>
<td>2</td>
<td>.67</td>
<td>.66</td>
<td>2.7107</td>
</tr>
<tr>
<td>2</td>
<td>25.26, 120.89, 366.27</td>
<td>3</td>
<td>.72</td>
<td>.84</td>
<td>3.2292</td>
</tr>
<tr>
<td>3</td>
<td>47.46, 350.2</td>
<td>2</td>
<td>.77</td>
<td>.46</td>
<td>3.8599</td>
</tr>
<tr>
<td>4</td>
<td>53.96, 315.42</td>
<td>2</td>
<td>.77</td>
<td>.46</td>
<td>3.9882</td>
</tr>
<tr>
<td>5</td>
<td>56.42, 72.09, 339.97</td>
<td>3</td>
<td>.87</td>
<td>.39</td>
<td>4.0328</td>
</tr>
<tr>
<td>6</td>
<td>99.57, 274.71</td>
<td>2</td>
<td>.92</td>
<td>.16</td>
<td>4.6009</td>
</tr>
<tr>
<td>7</td>
<td>100.31</td>
<td>1</td>
<td>.50</td>
<td>.50</td>
<td>4.6083</td>
</tr>
<tr>
<td>8</td>
<td>111.99, 263.47, 373.03</td>
<td>3</td>
<td>.85</td>
<td>.45</td>
<td>4.7184</td>
</tr>
<tr>
<td>9</td>
<td>125.48, 164.66, 303.98</td>
<td>3</td>
<td>.89</td>
<td>.33</td>
<td>4.8321</td>
</tr>
<tr>
<td>10</td>
<td>133.43, 177.38, 324.95, 364.63</td>
<td>4</td>
<td>.74</td>
<td>1.04</td>
<td>4.8936</td>
</tr>
<tr>
<td>11</td>
<td>192.68</td>
<td>1</td>
<td>.70</td>
<td>.30</td>
<td>5.2009</td>
</tr>
<tr>
<td>12</td>
<td>249.15, 324.47</td>
<td>2</td>
<td>.63</td>
<td>.74</td>
<td>5.5181</td>
</tr>
<tr>
<td>13</td>
<td>285.01</td>
<td>1</td>
<td>.64</td>
<td>.36</td>
<td>5.6525</td>
</tr>
<tr>
<td>14</td>
<td>379.43</td>
<td>1</td>
<td>.72</td>
<td>.28</td>
<td>5.9387</td>
</tr>
<tr>
<td>15</td>
<td>388.97</td>
<td>1</td>
<td>.69</td>
<td>.31</td>
<td>5.9635</td>
</tr>
<tr>
<td>16</td>
<td>395.25</td>
<td>1</td>
<td>.46</td>
<td>.54</td>
<td>5.9795</td>
</tr>
<tr>
<td>Totals</td>
<td></td>
<td>32</td>
<td>11.54</td>
<td>7.82</td>
<td>75.7873</td>
</tr>
</tbody>
</table>

From Equation (1) and Table 1, the adjustment procedure estimate of \(r(T) = r(400)\) is

\[
\hat{r}_{\text{adj}}(400) = \left( \frac{1}{400} \right) \left( N_x + \sum_{i=1}^{16} (1 - d_i^*)N_i \right)
\]

\[
= \frac{10 + 7.82}{400} = 0.04455
\]

Thus the adjustment procedure estimate of the system MTBF is

\[
\left\{\hat{r}_{\text{adj}}(400)\right\}^{-1} = \frac{400}{17.82} = 22.45
\]

Looking at Equation (40), we can see that the adjustment procedure estimate of system failure intensity after implementation of the fixes is simply \(\hat{\rho}_{GP}\), the estimated growth potential failure intensity. Thus

\[
\hat{\rho}_{GP} = \hat{r}_{\text{adj}}(400) = 0.04455
\]

Also, the estimate of the system growth potential MTBF is

\[
\hat{\rho}_{GP}^{-1} = \left\{\hat{r}_{\text{adj}}(400)\right\}^{-1} = 22.45
\]

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To obtain an estimate with less bias of the system’s failure intensity and corresponding MTBF at T=400 hours, after incorporation of fixes to the sixteen surfaced B-modes, we use the Crow/AMSAA model estimation equation (32). This projection is given by

$$\hat{\rho}_c(400) = \hat{\rho}_{GP} + \left( \frac{\hat{\beta}}{400} \right) \sum_{i=obs} d_i^*$$

$$= 0.04455 + \left( \frac{\hat{\beta}}{400} \right)(11.54) \quad (53)$$

The mle $\hat{\beta}$ is obtained from Equation (26), i.e.,

$$\hat{\beta} = \frac{m}{\sum_{i=1}^n \ln \left( \frac{T}{t_i} \right)} = \frac{m}{m \ln T - \sum_{i=1}^n \ln t_i}$$

$$= \frac{16}{16 \ln 400 - 75.7873} = 0.7970$$

Thus, by (53), the Crow/AMSAA projection for the system failure intensity, based on $\hat{\beta}$, is

$$\hat{\rho}_c(400) = 0.04455 + \left( \frac{0.7970}{400} \right)(11.54)$$

$$= 0.06754$$

The corresponding MTBF projection is

$$\left( \hat{\rho}_c(400) \right)^{-1} = 14.81$$

A nearly unbiased assessment of the system failure intensity, for $d_i^* = d_i$, can be obtained by using $\bar{\beta}_m$ instead of $\hat{\beta}$. Recall by (28),

$$\bar{\beta}_m = \left( \frac{m-1}{m} \right) \hat{\beta} = \left( \frac{15}{16} \right)(0.7970) = 0.7472$$

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By Equation (33), the projected system failure intensity based on $\bar{\rho}_m$ is

$$\bar{\rho}_c(400) = \bar{\rho}_{GP} + \frac{\bar{\rho}_m}{T} \sum_{i=1}^{\text{obs}} d_i^*$$

$$= 0.04455 + \left( \frac{0.7472}{400} \right)(11.54)$$

$$= 0.06611$$

The corresponding MTBF projection is

$$\left\{ \bar{\rho}_c(400) \right\}^{-1} = 15.13$$

As discussed in Section 4.3.5, we recommend basing the projected system failure intensity on $\bar{\rho}_c(T)$ which uses $\bar{\rho}_m$, but assess the projected system MTBF by using $\hat{\beta}$. Thus in this example we would recommend assessing the projected system failure intensity by

$$\bar{\rho}_c(400) = 0.06611$$

and the projected system MTBF by

$$\left\{ \bar{\rho}_c(400) \right\}^{-1} = 14.81$$

4.4 The AMSAA Maturity Projection Model (AMPM) – Continuous.

4.4.1 Introduction. The continuous version of the AMPM assumes the test duration is measured in a continuous scale such as time or miles. Throughout this section AMPM will refer to the continuous version of the model and we shall refer to time as the measure of test duration.

The AMPM addresses making reliability projections in several situations of interest. One case corresponds to that addressed by the Crow/AMSAA projection model introduced in [1] and discussed in Section 4.3. This is the situation in which all fixes to B-modes are implemented at the end of the current test phase, Phase I, prior to commencing a follow-on test phase, Phase II. The projection problem is to assess the expected system failure intensity at the start of Phase II. Another situation handled by the AMPM estimation procedure is the case where the reliability of the unit under test has been maturing over Test Phase I due to implemented fixes during Phase I. This case includes the situations where
(i) all surfaced B-modes in Test Phase I have fixes implemented within this test phase or

(ii) some of the surfaced B-modes are addressed by fixes within Test Phase I and the remainder are treated as delayed fixes, i.e., are fixed at the conclusion of Test Phase I, prior to commencing Test Phase II.

A third type of projection of interest involves projecting the system failure intensity at a future program milestone. This future milestone may occur beyond the commencement of the follow-on test phase.

All the above type of projections are based on the Phase I B-mode first occurrence times, whether the associated B-mode fix is implemented within the current test phase or delayed (but implemented prior to the projection time). In addition to the B-mode first occurrence times, the projections are based on an average fix effectiveness factor (FEF). This average is with respect to all the potential B-modes, whether surfaced or not. However, as in the Crow/AMSAA model, this average FEF is assessed based on the surfaced B-modes. For the AMPM model, the set of surfaced B-modes would typically be a mixture of B-modes addressed with fixes during the current test phase as well as those addressed beyond the current test phase.

In some instances, a reliability projection for a future milestone can be based on extrapolating a reliability growth tracking curve. Such a curve only utilizes cumulative failure times and does not use B-mode fix effectiveness factors. This is a valid projection approach provided it is reasonable to expect that the observed pattern of reliability growth will continue up through the milestone of interest. However, this pattern could change in a pronounced manner. Reasons for such a change include

(i) a change in the test environment;

(ii) a different level of future resources to analyze and implement effective corrective actions; and

(iii) jumps in reliability due to delayed fixes.

If extrapolating the current tracking curve is not deemed suitable due to considerations such as above, the AMPM projection methodology may be useful. Unlike assessments based on the tracking model, the AMPM assessments are independent of the fix discipline, as long as the fixes are implemented prior to the projection milestone date of interest. Unlike the reliability growth tracking model in [2], the AMPM (as well as the Crow/AMSAA projection model) utilizes a non-homogeneous Poisson process with regard to the number of distinct B-modes that occur by test duration t. The associated pattern of B-mode first occurrence times is not dependent on the corrective action strategy, under the assumption that corrective actions are not inducing new B-modes to occur. Thus the AMPM assessment procedure is not upset by jumps in reliability due to delayed groups of fixes. In contrast, reliability growth tracking curve methodology
utilizes the pattern of cumulative failure times. Such a pattern is sensitive to the corrective action strategy. Thus a reliability growth tracking curve model may not be appropriate for fitting failure data or for extrapolating due to a corrective action strategy that is not compatible with the model.

Note that AMPM reliability projections for a future milestone would be optimistic if corrective actions beyond the current test phase were less effective than the average FEF assessment based on B-modes surfaced through the current test phase. Also, a change in the future testing environment could result in a new set of potential failure modes or affect the rates of occurrence of the original set of failure modes. Either of these circumstances would tend to degrade the accuracy of the AMPM reliability projection.

Another instance in which a reliability projection model would be useful is when the current test phase contains a number of design configurations of the units under test due to incorporation of reliability fixes during the test phase. If there is a lack of fit of the reliability growth tracking model over these configurations then the tracking model should not be used to assess the reliability of the latest configuration or for extrapolation to a future milestone. Such a lack of fit may be due to the corrective action process, i.e., when the fixes are implemented and their effectiveness. As pointed out earlier, the AMPM, unlike a tracking model, is insensitive to any nonsmoothness in the expected number of failures versus test time that results from the timing or effectiveness of corrective actions. Thus in such a situation, program management may wish to use a projection method such as the AMPM to assess the reliability of the current configuration or to project the expected reliability at a future milestone.

As discussed in [3], the AMPM can also be used to construct a useful reliability maturity metric. This metric is the fraction of the expected initial system B-mode failure intensity, $\lambda_0$, surfaced by test duration $t$. By this we mean the expected fraction of $\lambda_0$ due to B-modes surfaced by $t$. This concept will be expanded upon in a later subsection.

Prior to presenting the model equations and estimation procedures, we shall list the associated notation and assumptions.

### 4.4.2 AMPM Notation and Assumptions.

**Notation:**

$\mu_d$  
Arithmetic average of the $d_i$, i.e., $\left(\frac{1}{K}\sum_{i=1}^{K} d_i\right)$

$\alpha, \beta$  
Parameters for gamma density function, where $\alpha > -1$ and $\beta > 0$ (subscripted by $K$ or $\infty$ where required for clarity).

$\alpha!$  
Denotes the integral $\int_0^\infty x^\alpha e^{-x} \, dx$ for $\alpha > -1$. 

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\( \Lambda \)  
Gamma random variable.

\( \Psi \)  
Moment generating function for \( \Lambda \).

\( \Gamma(\alpha, \beta) \)  
Denotes gamma random variable with parameters \( \alpha > -1, \beta > 0 \).

\( f_{\lambda} \)  
Denotes density function for \( \Lambda \sim \Gamma(\alpha, \beta) \), where

\[
   f_{\lambda}(\lambda) = \frac{\lambda^{\alpha-1} e^{-\lambda/\beta}}{\alpha! \beta^\alpha} \quad \text{for } \lambda > 0;
\]

\[
   = 0 \quad \text{elsewhere}
\]

\( \Delta = (\Lambda_1, \ldots, \Lambda_K) \)  
Random sample of size \( K \) from \( \Gamma(\alpha, \beta) \).

\( \hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_K) \)  
Realization of \( \Delta \).

\( \lambda_{\beta, K} \)  
Expected value of \( \sum_{i=1}^{K} \Lambda_i \).

\( \lambda_{\beta, \infty} = \lim_{K \to \infty} \lambda_{\beta, K} \)

\( \mu(t; \hat{\lambda}) \)  
Expected number of distinct B-modes conditioned on \( \Delta = \hat{\lambda} \).

\( h(t; \hat{\lambda}) \)  
Expected rate of occurrence of B-modes given \( \Delta = \hat{\lambda} \).

\( h(t) \)  
Unconditional expected B-mode rate of occurrence.

\( r(t; \hat{\lambda}) \)  
System failure intensity after fixes to B-modes surfaced by t have been implemented, conditioned on \( \Delta = \hat{\lambda} \).

\( \rho(t; \hat{\lambda}) \)  
Expected value of \( r(t; \hat{\lambda}) \) with respect to random first occurrence times of B-modes.

\( \rho(t) \)  
Expectation of \( \rho(t; \Delta) \) with respect to \( \Delta \).

\( I_i(t) \)  
Equals 1 if B-mode \( i \) occurs by \( t \), equals 0 otherwise.
$u(t; \hat{\lambda})$  
Failure intensity at time $t$ due to unsurfaced B-modes, conditioned on $\Delta = \hat{\lambda}$.

$\lambda(t)$  
Unconditional expected failure intensity due to set of B-modes surfaced by $t$, in absence of any fixes.

$\Theta(t)$  
Expected fraction of $\hat{\lambda}_{b,k}$ surfaced as a function of $t$.

$t_i$  
Time of first occurrence of B-mode $i$.

$\zeta = (t_1, \ldots, t_m)$

$L(m, t, \hat{\lambda})$  
Likelihood function for the test data $(m, t)$ given $\Delta = \hat{\lambda}$.

$L(m, t)$  
Expectation of $L(m, t, \Delta)$.

$\ln$  
Natural logarithm (base "e").

$Z$  
$\ln\{L(m, t)\}$

$\nu_K$  
$(\alpha, \beta, K)$

$\nu_K$  
$(\hat{\alpha}_K, \hat{\beta}_K, K)$

obs  
Set of indices associated with $m$ observed B-modes.

$K_0$  
Greatest lower bound for set of $K$-values for which AMPM mle's are well defined.

$K_{BM}$  
IBM model mle of $K$.

$\Delta$  
Defined to be.

Additional Assumptions for AMPM – Continuous

- The time to first occurrence is exponentially distributed for each failure mode.
• For \(i = 1, 2, \ldots, K\), the effectiveness of a fix associated with B-mode \(i\) is independent of the mode’s initial rate of occurrence \(\lambda_i\).

• The B-mode initial rates of occurrence \((\lambda_1, \ldots, \lambda_K)\) constitute the realization of a random sample \((\Lambda_1, \ldots, \Lambda_K)\) from a gamma distribution with density \(f_{\alpha\beta}\). This models mode-to-mode variation in the B-mode initial failure rates. That is, we assume the \(\Lambda_i (i = 1, \ldots, K)\) are independent and identically distributed (IID) random variables, where \(\Lambda_i \sim \Gamma(\alpha, \beta)\).

### 4.4.3 AMPM Development

The AMPM provides a procedure for assessing the system failure intensity \(r(t; \Lambda)\). Recall \(r(t; \Lambda)\) denotes the system failure intensity after fixes to all B-modes surfaced by test time \(t\) have been implemented.

Note \(\Lambda = (\lambda_1, \ldots, \lambda_K)\) denotes the initial B-mode rates of occurrence. In particular, consider B-mode \(i\). If this mode does not occur by \(t\) then its rate of occurrence at \(t\) is still \(\lambda_i\). However, if B-mode \(i\) occurs by \(t\) then, by our definition of \(r(t; \Lambda)\), the contribution of this mode to \(r(t; \Lambda)\) is only \(1 - d_i \lambda_i\) due to the implemented fix (or fixes) to mode \(i\) by \(t\). We may conveniently mathematically express the contribution of B-mode \(i\) to \(r(t; \Lambda)\) by

\[
(1 - d_i I_i(t)) \lambda_i
\]

Thus

\[
r(t; \Lambda) = \lambda_A + \sum_{i=1}^{K} (1 - d_i I_i(t)) \lambda_i
\]

\[= \lambda_A + \sum_{i=1}^{K} \lambda_i - \sum_{i=1}^{K} d_i \lambda_i I_i(t)\]  \hspace{1cm} (2)

As in the Crow/AMSAA model, the AMPM assesses the system failure intensity \(r(t; \Lambda)\) by an assessment of the expected value of \(r(t; \Lambda)\), i.e. \(\rho(t; \Lambda) = E(r(t; \Lambda))\). Note by (2) we have

\[
\rho(t; \Lambda) = E(r(t; \Lambda))
\]

\[= \lambda_A + \sum_{i=1}^{K} \lambda_i - \sum_{i=1}^{K} d_i \lambda_i E(I_i(t))\]  \hspace{1cm} (3)

In Appendix D, Annex 1 we show,
\[ E[l_i(t)] = 1 - e^{-\lambda_i t} \]  

(4)

where the expectation is with respect to the time of first occurrence of B-mode \( i \). This yields

\[ \rho(t; \bar{A}) = \lambda_i + \sum_{i=1}^{K} (1 - d_i) \lambda_i + \sum_{i=1}^{K} d_i \lambda_i e^{-\lambda_i t} \]  

(5)

In Section 4.3 (where the argument \( \bar{A} \) was suppressed) it was noted that the Crow/AMSAA model approximates \( \rho(t; \bar{A}) \) by

\[ \rho_c(t; \bar{A}) = \lambda_i + \sum_{i=1}^{K} (1 - d_i) \lambda_i + \mu \mu_c(t; \bar{A}) \]  

(6)

with

\[ h_c(t; \bar{A}) = uvt^{-1} \]  

(7)

for positive constants \( u, v \). This form for the expected rate of occurrence of new B-modes corresponds to approximating the expected number of distinct B-modes occurring over \([0, t]\) by

\[ \mu_c(t; \bar{A}) = ut^r \]  

(8)

Recall the Crow/AMSAA procedure estimates the constants \( u, v \) by the mle statistics based on the B-mode first occurrence times observed during Test Phase I, i.e., \([0, T]\). The summation term in (6) is assessed as

\[ \sum_{i \in \text{obs}} (1 - d_i) \frac{N_i}{T} \]  

(9)

where \( d_i \) is the assessed fix effectiveness factor for observed B-mode \( i \), and \( N_i \) is the number of occurrences of failures during \([0, T]\) attributed to B-mode \( i \). Note in the Crow/AMSAA procedure all fixes are assumed to be delayed to the end of the period \([0, T]\). Under this assumption \( \frac{N_i}{T} \) is an unbiased estimate of \( \lambda_i \). However, if fixes to B-modes are implemented prior to the end of this period (9) may not be an adequate assessment of \( \sum_{i=1}^{K} (1 - d_i) \lambda_i \).

The AMPM does not attempt to assess \( \rho(t; \bar{A}) \) by estimating each \( \lambda_i \). Instead the AMPM approach is to view \((\lambda_1, \cdots, \lambda_k)\) as a realization of a random sample.
\( \Lambda = (\Lambda_1, \cdots, \Lambda_K) \) from the gamma random variable \( \Gamma(\alpha, \beta) \). This allows one to utilize all the B-mode times to first occurrence observed during Test Phase I to estimate the gamma parameters \( \alpha, \beta \). Thus in place of directly assessing \( \rho(t; \underline{\Lambda}) \), the AMPM uses estimates of \( \alpha \) and \( \beta \) to assess the expected value of \( \rho(t; \Lambda) \) where

\[
\rho(t; \Lambda) = \lambda_d + \sum_{i=1}^{K} (1 - d_i) \Lambda_i + \sum_{i=1}^{K} d_i \Lambda_i e^{-\lambda_i t} \tag{10}
\]

This assessed value is then taken as the AMPM assessment of the system failure intensity after fixes to all B-modes surfaced over \([0, t]\) have been implemented. This approach does away with the need to estimate individual \( \lambda_i \). Trying to adequately estimate individual \( \lambda_i \) could be particularly difficult in the case where many fixes are implemented prior to the end of the period \([0, t]\).

From Equation (10) we see that the expected value of \( \rho(t; \Lambda) \) with respect to the random sample \( \Lambda \), denoted by \( \rho(t) \), is given by

\[
\rho(t) = \lambda_d + \sum_{i=1}^{K} (1 - d_i) E(\Lambda_i) + \sum_{i=1}^{K} d_i E(\Lambda_i e^{-\lambda_i t}) \tag{11}
\]

Recall the \( \Lambda_i \) are IID with \( \Lambda_i \sim \Lambda \). Thus \( E(\Lambda_i) = E(\Lambda) \) and \( E(\Lambda_i e^{-\lambda_i t}) = E(\Lambda e^{-\lambda t}) \) for \( i = 1, \cdots, K \). After rearranging terms and replacing \( \sum_{i=1}^{K} d_i \) by \( K \mu_d \), \( E(\Lambda_i) \) by \( E(\Lambda) \), and \( E(\Lambda_i e^{-\lambda_i t}) \) by \( E(\Lambda e^{-\lambda t}) \) we arrive at

\[
\rho(t) = \lambda_d + (1 - \mu_d) \{ K E(\Lambda) \} + \mu_d \{ K E(\Lambda e^{-\lambda t}) \} \tag{12}
\]

Next note

\[
\lambda_{g,K} = E \left( \sum_{i=1}^{K} \Lambda_i \right) = K E(\Lambda) \tag{13}
\]

Thus we can express \( \rho(t) \) by

\[
\rho(t) = \lambda_d + (1 - \mu_d) \lambda_{g,K} + \mu_d \left[ K E(\Lambda e^{-\lambda t}) \right] \tag{14}
\]

To interpret the term \( K E(\Lambda e^{-\lambda t}) \) in (14) we first note that in Appendix D, Annex 1, it is shown that
\[ \mu(t; \Delta) = \sum_{i=1}^{k} (1 - e^{-\lambda_i t}) \]

Thus the expected rate of occurrence of new B-modes at \( t \) given \( \Delta \), is

\[ h(t; \Delta) = \frac{d \mu(t; \Delta)}{dt} = \sum_{i=1}^{k} \lambda_i e^{-\lambda_i t} \]

Consider the average (i.e., expected) value of \( h(t; \Delta) = \sum_{i=1}^{k} \lambda_i e^{-\lambda_i t} \) over all possible random samples \( \Delta = (\Lambda_1, \ldots, \Lambda_K) \), where \( \Lambda_i \sim \Lambda \) for \( i = 1, \ldots, K \). We obtain

\[ E(h(t; \Delta)) = \sum_{i=1}^{k} E(\lambda_i e^{-\lambda_i t}) = K E(\Lambda e^{-\Lambda t}) \]  

(15)

Let \( h(t) = E(h(t; \Lambda)) \). Thus \( h(t) \) is the unconditional expected rate of occurrence of new B-modes at test time \( t \) averaged over all possible random samples \( \Delta \). By (14) and (15) we have

\[ \rho(t) = \lambda_A + (1 - \mu_d) \lambda_{B,K} + \mu_d h(t) \]  

(16)

This expression for \( \rho(t) \) is similar in form to the Crow/AMSAA approximation to \( \rho(t; \Delta) \) given in Equation (14):

\[ \rho_c(t; \Delta) = \lambda_A + \sum_{i=1}^{k} (1 - d_i) \lambda_i + \mu_d h_c(t) \]

where reference to \( \Delta \) was suppressed in the notation.

The expression in (16) for the expected system failure intensity after incorporation of B-mode fixes is actually quite appealing to one’s intuition if put in a slightly different form. To arrive at this form we shall simply subtract and add the term \( h(t) \) on the right hand side of Equation (16). Doing this we can express \( \rho(t) \) by

\[ \rho(t) = \lambda_A + (1 - \mu_d) \lambda_{B,K} + h(t) \]  

(17)

Now we see that \( \rho(t) \) is the sum of three failure intensities. The first is simply the constant failure intensity due to the A-modes. To consider the second failure intensity we shall first consider \( h(t) \). We have shown that this term is the expected rate of occurrence of new B-modes at test time \( t \) averaged over the random samples \( \Delta \). Additionally, \( h(t) \) is the expected failure intensity contribution to \( \rho(t) \) due to the set of B-modes that have
not been surfaced by $t$. To see this, first note that the failure intensity at time $t$, conditioned on $\Delta = \Delta$, due to unsurfaced B-modes is $u(t; \Delta)$ where

$$u(t; \Delta) = \sum_{i=1}^{K} \{1 - I_i(t)\} \lambda_i$$  \hspace{1cm} (18)

Recall by (4),

$$E[I_i(t)] = 1 - e^{-\Delta t}$$

with respect to the first occurrence of B-mode $i$. Thus by (18) we have

$$E[u(t; \Delta)] = \sum_{i=1}^{K} \lambda_i - \sum_{i=1}^{K} \lambda_i E[I_i(t)]$$

$$= \sum_{i=1}^{K} \lambda_i e^{-\Delta t} = h(t; \Delta)$$  \hspace{1cm} (19)

It immediately follows from (19) that $h(t)$ is the unconditional expected failure intensity due to the set of unsurfaced B-modes at time $t$, since $h(t) = E(h(t; \Delta))$.

Finally, we consider the second term of $\rho(t)$ in (17). In the absence of any fixes, the sum of $h(t)$ and the unconditional expected failure intensity due to the set of B-modes surfaced by $t$, denoted by $s(t)$, must equal $\lambda_{s,K}$. Thus $s(t) = \lambda_{s,K} - h(t)$. If we implement fixes to the B-modes surfaced by $t$ with an average FEF equal to $\mu_d$, then the residual expected failure intensity due to the set of surfaced B-modes would be

$$(1 - \mu_d) s(t) = (1 - \mu_d) (\lambda_{s,K} - h(t))$$  \hspace{1cm} (20)

In the above equations we can replace $\lambda_{s,K}$ by $h(0)$ since at $t=0$ all B-modes are unsurfaced. Thus

$$h(0) = \lambda_{s,K}$$  \hspace{1cm} (21)

As in Section 4.3, we call the residual expected failure intensity approached by $\rho(t)$ as $t$ tends towards infinity the growth potential failure intensity, denoted by $\rho_{GP}$.

Since $\lim_{t \to \infty} h(t) = 0$ we have

$$\rho_{GP} = \lambda_s + (1 - \mu_d) \lambda_s$$  \hspace{1cm} (22)
Note this expression has a form similar to that for the growth potential in the Crow/AMSAA model. The quantity \( \rho_{\text{op}}^{1} \) is called the growth potential MTBF. The growth potential for the AMPM is used in the same way as indicated in Section 4.3 for the Crow/AMSAA model.

Another useful quantity is the expected fraction of the system expected initial B-mode failure intensity, \( \lambda_{b} \), surfaced as a function of test time \( t \). We shall let \( \theta(t) \) denote this quantity. Thus, by definition of \( s(t) \), we have

\[
\theta(t) = \frac{s(t)}{\lambda_{b,K}} = \frac{\lambda_{b,K} - h(t)}{\lambda_{b,K}}
\]

(23)

Note that \( \theta(t) \) is independent of the corrective action process. By this we mean that \( \theta(t) \) does not depend on when fixes are implemented nor on how effective they are.

The function \( \theta(t) \) can usefully serve as a measure of system maturity. Observe that for a test of duration \( t \), no matter how effective our fixes are, we can only expect to eliminate at most a fraction equal to \( \theta(t) \) of the expected B-mode contribution to the initial system failure intensity. Thus low values of \( \theta(t) \) would indicate additional testing is required to surface a set of B-modes that account for a significant part of \( \lambda_{b} \). A high value for \( \theta(t) \) could indicate that further testing is not cost effective. Resources would be better expended toward formulating and implementing corrective actions for the surfaced B-modes. As part of a reliability growth plan it would be useful to specify goals for \( \theta(t) \) at several program milestones.

Next we shall express the key AMPM reliability projection quantities in terms of \( K \) and the gamma parameters \( \alpha \) and \( \beta \). By Appendix D, Annex 2, we have

\[
\lambda_{b,K} = K\beta(\alpha + 1)
\]

(24)

\[
\mu(t) = K\left\{(1 + \beta t)^{-(\alpha - 1)}\right\}
\]

(25)

\[
h(t) = \frac{K\beta(\alpha + 1)}{(1 + \beta t)^{\alpha - 2}} - \frac{d\mu(t)}{dt}
\]

(26)

\[
\rho(t) = \lambda_{a} + (1 - \mu_{d})K\beta(\alpha + 1) + \frac{\mu_{d}K\rho(\alpha + 1)}{(1 + \beta t)^{\alpha - 2}}
\]

(27)

and

\[
\theta(t) = 1 - (1 + \beta t)^{-(\alpha - 1)}
\]

(28)
Utilizing Equation (24) for $\lambda_{b,k}$ we can also express $h(t)$ and $\rho(t)$ as follows:

$$h(t) = \frac{\lambda_{b,k}}{(1 + \beta t)^{a+2}} \quad (29)$$

and

$$\rho(t) = \lambda_i + (1 - \mu_d)\lambda_{b,k} + \frac{\mu_d \lambda_{b,k}}{(1 + \beta t)^{a+2}} \quad (30)$$

In the next section, we shall consider the behavior of the AMPM as $K$ increases. Limiting expressions for the AMPM quantities in (24) through (30) will be obtained as $K \to \infty$ under natural assumptions about $\lambda_{b,k}$ and $\beta = \beta_K$. Then parameter estimation procedures will be specified for the finite $K$ AMPM and the limiting parameters as $K \to \infty$.

4.4.4 Limiting Behavior of AMPM. We shall now consider the limiting behavior of the AMPM as $K$ increases. To do so we first define step processes $\{X_{K,i}(t), 0 \leq t < \infty\}$ for $i = 1, \ldots, K$ where

$$X_{K,i}(t) = \begin{cases} 1 & \text{if } B - \text{mode } i \text{ occurs by } t \\ 0 & \text{otherwise} \end{cases}$$

Note

$$\Pr(X_{K,i}(t) \geq 2) = 0 \quad (31)$$

and

$$\Pr(X_{K,i}(t) = 1) = 1 - \Pr(X_{K,i}(t) = 0) \quad (32)$$

Thus to complete our definition of these processes, we need only specify $\Pr(X_{K,i}(t) = 0)$. To keep the definition of these processes consistent with the AMPM assumptions we define

$$\Pr(X_{K,i}(t) = 0) = \int_0^\infty e^{-\alpha} f_\lambda(x) dx \quad (33)$$

where $\alpha \sim \Gamma(\alpha, \beta)$ and $f_\lambda$ is the previously defined gamma density function with $\alpha = \alpha_K$ and $\beta = \beta_K$. Note $X_{K,i}(t)$ is the unconditional AMPM indicator function for B-mode $i$ corresponding to the earlier defined conditional indicator function $I_i(t)$ where
\[ \Pr(I_i(t) = 0) = e^{-\lambda t} \]

and subscript \(K\) was suppressed. By (33) and Appendix D, Annex 2

\[ \Pr(X_{K,i}(t) = 0) = E(e^{-\lambda t}) = \Psi(-t) = (1 + \beta \kappa t)^-(\alpha + 1) \quad (34) \]

From (32) and (34) we obtain

\[ \mu_K(t) \triangleq E\left[ \sum_{i=1}^{K} X_{K,i}(t) \right] \]

\[ = \sum_{i=1}^{K} \Pr(X_{K,i}(t) = 1) = K - K(1 + \beta \kappa t)^-(\alpha + 1) = \mu(t) \quad (35) \]

for \((\alpha_k, \beta_k) = (\alpha, \beta)\). Thus the AMPM step processes \( \{X_{K,i}(t)\}, 0 \leq t < \infty \), \(1 \leq i \leq K\), give rise to our previously developed AMPM.

To investigate the behavior of our projection model as \(K\) increases, we must specify the limiting behavior of \(\alpha_k\) and \(\beta_k\). Since \(\beta_k\) is simply a scale factor for test time \(t\) it is reasonable to keep \(\beta_k\) fixed, say \(\beta_k = \beta_\infty \in (0, \infty)\). Recall by (24),

\[ \lambda_{B,K} = K \beta_k (\alpha_k + 1) \].

Regardless of the value of \(K\), \(\lambda_{B,K}\) represents the unconditional expected B-mode contribution to the initial system failure intensity. Thus it is natural to let \(K \beta_k (\alpha_k + 1) = \lambda_{B,\infty} \in (0, \infty)\) for all \(K\). Actually, to obtain our results for the limiting behavior of the AMPM we need only insist that

\[ \lim_{K \to \infty} \beta_k = \beta_\infty \in (0, \infty) \quad (36) \]

and

\[ \lim_{K \to \infty} K \beta_k (\alpha_k + 1) = \lambda_{B,\infty} \in (0, \infty) \quad (37) \]

We shall simply denote \(\beta_\infty\) and \(\lambda_{B,\infty}\) by \(\beta\) and \(\lambda_{B}\), respectively. Since \(\alpha_k + 1 \geq 0\), (36) and (37) imply

\[ \lim_{K \to \infty} \alpha_k = -1 \quad (38) \]

Let \(X_K(t)\) be the supposition of the independent step processes \(X_{K,i}(t)\), i.e.
\[ X_K(t) \triangleq \sum_{i=1}^{c} X_{K,i}(t) \]  

(39)

It is demonstrated in [4] that the stochastic process \( \{ X_K(t), 0 \leq t < \infty \} \) converges to a nonhomogeneous Poisson process (NHPP) with mean value function \( \mu_\infty(t) \) as \( K \to \infty \), where

\[ \mu_\infty(t) = \left( \frac{\lambda_\infty}{\beta} \right) \ln(1 + \beta t) \]  

(40)

This result suggests that for complex systems or subsystems, we can expect our AMPM process \( \{ X_K(t), 0 \leq t < \infty \} \) to behave like a NHPP \( \{ X_\infty(t), 0 \leq t < \infty \} \) where \( X_\infty(t) \) is the number of distinct B-modes that occur by \( t \) and \( E\{ X_\infty(t) \} = \mu_\infty(t) \) given in (40).

We can now relate the key AMPM reliability projection quantities in (24) through (28) which depend on \( K \) to the corresponding NHPP quantities. To do so we shall subscript the AMPM quantities by \( K \) and the NHPP quantities by \( \infty \). Thus, for example, by (24) and limit condition (37) we have

\[ \lim_{K \to \infty} \lambda_{\beta,K} = \lambda_{\beta,\infty} \in (0, \infty) \]  

(41)

(where we also denote \( \lambda_{\beta,\infty} \) simply by \( \lambda_{\beta} \)). By (26) we also have

\[ \mu_K(t) = \int_{0}^{t} \int_{0}^{K \beta_k^k \left( \alpha_k + 1 \right)} \frac{dz}{(1 + \beta_k z)^{x_k + 2}} \]  

Thus

\[ \lim_{K \to \infty} \mu_K(t) = \int_{0}^{t} \lim_{K \to \infty} \left[ \frac{K \beta_k^k \left( \alpha_k + 1 \right)}{(1 + \beta_k z)^{x_k + 2}} \right] dz \]

By (36) through (38) and (40) this yields

\[ \lim_{K \to \infty} \mu_K(t) = \int_{0}^{t} \frac{\lambda_\infty dz}{1 + \beta z} = \left( \frac{\lambda_\infty}{\beta} \right) \ln(1 + \beta t) = \mu_\infty(t) \]  

(42)

Again by (26), (36) through (38) and (40), we obtain

\[ \lim_{K \to \infty} h_K(t) = \lim_{K \to \infty} \frac{K \beta_k^k \left( \alpha_k + 1 \right)}{(1 + \beta_k t)^{x_k + 2}} \]
\[
\frac{\dot{\lambda}_g}{1 + \beta t} = \frac{d \mu_x(t)}{dt} = h_x(t)
\]  

(43)

By (27), (36) through (38) and (43) we arrive at

\[
\lim_{K \to \infty} \rho_K(t) = \lim_{K \to \infty} \left\{ \dot{\lambda}_A + (1 - \mu_d) K \beta_K (\alpha_K + 1) + \frac{\mu_d K \beta_K (\alpha_K + 1)}{(1 + \beta_K t)^{\alpha_K + 2}} \right\}
\]

= \lambda_A + (1 - \mu_d) \dot{\lambda}_g - \frac{\mu_d \dot{\lambda}_g}{1 + \beta t} = \lambda_A + (1 - \mu_d) \dot{\lambda}_g + \mu_d h_x(t)

= \rho_x(t)
\]

(44)

Additionally, by (22) and (41) we have

\[
\lim_{K \to \infty} \rho_{GP,K} = \lim_{K \to \infty} \left\{ \dot{\lambda}_A + (1 - \mu_d) \lambda_{B,K} \right\}
\]

= \lambda_A + (1 - \mu_d) \lambda_A = \rho_{GP,\infty}
\]

(45)

Finally, by (28), (36) and (38) we deduce

\[
\lim_{K \to \infty} \theta_x(t) = \lim_{K \to \infty} \left\{ \frac{1}{1 + \beta_K t} \right\} = \frac{\beta t}{1 + \beta t}
\]

Thus by (43) we conclude

\[
\lim_{K \to \infty} \theta_x(t) = \frac{\beta t}{1 + \beta t} = \frac{\dot{\lambda}_g - h_x(t)}{\lambda_g} = \theta_x(t)
\]

(46)

4.4.5 Estimation Procedure for AMPM. In this section we shall specify the procedures to estimate key AMPM parameters and reliability measures expressed in terms of these parameters. Estimation equations will be given for the finite K and NHPP variants of the continuous AMPM. The model parameter estimators are mle's. Statistical details and further discussion of the estimation procedures are provided in Appendix D, Annex 3.

Our parameter estimates are written in terms of the following data: \( m \) = number of distinct B-modes that occur over a test period of length \( T \), \( \tau = (t_1, \cdots, t_m) \) where \( 0 < t_1 \leq t_2 \leq \cdots \leq t_m \leq T \) are the first occurrence times of the \( m \) observed B-modes, and \( n_x \) = number of A-mode failures that occur over test period \( T \). We shall denote an estimate of a model parameter or expression by placing the symbol "^" over the quantity.
The finite $K$ AMPM estimates are based on a specified value of $K$. If we hold the test data constant and let $K \to \infty$ we obtain AMPM projection estimates that are appropriate for complex subsystems or systems that typically have many potential B-modes. The AMPM limit estimating equations are derived in Appendix D, Annex 3. These equations can also be obtained from mle equations for the NHPP associated with the AMPM. This process was discussed in Section 4.4.4 and has the mean value function given by Equation (40).

Recall $\alpha_k, \beta_k$ are the gamma parameters for the AMPM where it is assumed the $K$ initial B-mode failure rates are realized values of a random sample from a gamma random variable $\Gamma(\alpha_k, \beta_k)$.

The mle for $\beta_k$ is $\hat{\beta_k}$ where

$$
K = \left( \sum_{i=1}^{m} \ln \frac{1 + \hat{\beta_k} T_i}{1 + \beta_k T_i} \right) \left( \sum_{i=1}^{m} \frac{1}{1 + \hat{\beta_k} T_i} \right) - \left( \frac{m \hat{\beta_k}}{\hat{\alpha_k} T} \right) \sum_{i=1}^{m} \frac{T - t_i}{1 + \beta_k T_i} \left( \ln \frac{1 + \hat{\beta_k} T}{1 + \beta_k T} \right) \left( \sum_{i=1}^{m} \frac{1}{1 + \beta_k T_i} \right) - \left( \frac{m \hat{\beta_k}}{\hat{\alpha_k} T} \right) T
$$

The mle for $\alpha_k$ is $\hat{\alpha_k}$ where $\hat{\alpha_k}$ can be easily obtained from $\hat{\beta_k}$ and either equation below. These equations are the maximum likelihood equations for $\alpha_k$ and $\beta_k$ respectively (see Appendix D, Annex 3):

$$
(\hat{\alpha_k} + 1)^{-1} = m^{-1} \left[ K \ln \left( 1 + \hat{\beta_k} T \right) - \sum_{i=1}^{m} \ln \left( \frac{1 + \hat{\beta_k} T_i}{1 + \beta_k T_i} \right) \right] \tag{48}
$$

$$
\hat{\alpha_k} + 1 = \frac{m \hat{\beta_k}}{(K - m) T} + \sum_{i=1}^{m} \frac{t_i}{1 + \beta_k T_i} \tag{49}
$$
Using \( \left( \alpha_K, \beta_K, K \right) \) we can estimate all our finite K AMPM quantities where the A-mode failure rate \( \lambda_A \) is estimated by \( \hat{\lambda}_A = \frac{n_A}{T} \) and the average B-mode fix effectiveness factor \( \mu_d \) is assessed as

\[
\mu_d = \frac{1}{m} \sum_{i=0}^{\infty} \lambda_i
\]

In (50), the assessment \( d_i \) of the fix effectiveness factor (FEF) for observed B-mode \( i \) will often be based largely on engineering judgement. The value of \( d_i \) should reflect several considerations: (1) How certain we are that the problem has been correctly identified; (2) the nature of the fix, e.g., its complexity; (3) past FEF experience and (4) any software testing (including assembly level testing).

Note the left-hand side of Equation (47) requires a value for \( K \) before we can numerically solve for \( \hat{\beta}_K \). In practice we do not know the value of \( K \). We could attempt to use the data \((m, t)\) to statistically estimate \( K \). However, graphs presented in the next section illustrate the difficulty in obtaining a reasonable estimate for \( K \) even for a large data set that appears to fit the model well. Thus we prefer to take the point of view that we should not attempt to statistically assess \( K \). However, by conducting a standard failure modes and effects criticality analysis (FMECA), we can place a lower bound on \( K \). Our experience with the AMPM indicates that if \( K \) is substantially higher than \( m \), say, e.g., \( K \geq 10m \), then our AMPM projection quantities will be insensitive to the value of \( K \). We believe for a complex system or subsystem it will often be the case that \( K \geq 10m \) or at least the unknown value of \( K \) will be \( 10m \) or higher. The factor 10 may be larger than necessary. We suggest exercising the finite K AMPM with several plausible lower bound values for \( K \) and comparing the associated projections with those obtained in the limit as \( K \to \infty \). This is illustrated for a data set in the next section.

To obtain the limiting AMPM projection model estimates consider the sequence of finite K AMPM estimates \( \left( \hat{\beta}_K \right)_{K > K_0} \) where we assume \( \hat{\beta}_K \) satisfies Equation (47) for each \( K > K_0 \). In Appendix D, Annex 3 it is shown that

\[
\beta_\infty = \lim_{K \to \infty} \hat{\beta}_K = (0, \infty)
\]

is a finite positive value. Moreover, it is demonstrated that
\[
\left\{ \ln \left(1 + \hat{\beta}_\infty T \right) \right\} \sum_{i=1}^{n} \frac{1}{1 + \hat{\beta}_\infty t_i} - \frac{m \beta_\infty T}{1 + \beta_\infty T} = 0 \tag{52}
\]

It is also shown that

\[
\hat{\alpha} = \lim_{K \to \infty} \hat{\alpha}_K = -1 \tag{53}
\]

where for each \( \hat{\beta}_K, K > K_0, \hat{\alpha}_K \) satisfies Equation (48) (or Equation (49)). The limiting AMPM estimates \( \hat{\beta}_\infty \) and \( \hat{\lambda}_\infty \), given below in Equation (55), can be shown to be mle's for parameters \( \beta_\infty \) and \( \lambda_{\beta, \infty} \). Recall these parameters define the NHPP discussed in Section 4.4.4 whose mean value function is given in Equation (40).

For ease of reference, the finite K AMPM and limiting AMPM estimates for key projection model quantities are listed below and indexed by K and \( \infty \), respectively:

\[
\hat{\lambda}_K = K \hat{\beta}_K \left( \hat{\alpha}_K + 1 \right) \tag{54}
\]

\[
\hat{\lambda}_\infty = \frac{m \beta_\infty}{\ln \left(1 + \beta_\infty T \right)} \tag{55}
\]

\[
\hat{\mu}_K(t) = K \left[ 1 - \left(1 + \hat{\beta}_K t \right)^{-\left(\hat{\alpha}_K + 1\right)} \right] \tag{56}
\]

\[
\hat{\mu}_\infty(t) = \left( \frac{\hat{\lambda}_{\beta, \infty}}{\beta_\infty} \right) \ln \left(1 + \beta_\infty t \right) \tag{57}
\]

\[
\hat{h}_K(t) = \frac{\hat{\lambda}_K}{\left(1 + \hat{\beta}_K t \right)^{\hat{\alpha}_K + 1}} \tag{58}
\]

\[
\hat{h}_\infty(t) = \frac{\hat{\lambda}_{\beta, \infty}}{1 + \beta_\infty t} \tag{59}
\]
\[ \hat{\rho}_{GP.K} = \hat{\lambda}_A + (1 - \mu_d^*) \hat{\lambda}_{B,K} \]  
(60)

\[ \hat{\rho}_{GP,m} = \hat{\lambda}_A + (1 - \mu_d^*) \hat{\lambda}_{B,m} \]  
(61)

\[ \hat{\rho}_K(t) = \hat{\rho}_{GP.K} + \mu_d^* \hat{h}_K(t) \]  
(62)

\[ \hat{\rho}_\infty(t) = \hat{\rho}_{GP.m} + \mu_d^* \hat{h}_\infty(t) \]  
(63)

\[ \hat{\theta}_K(t) = 1 - \left(1 + \hat{\beta}_K t \right)^{-\frac{\omega_0^*}{\alpha_0^*}} \]  
(64)

\[ \hat{\theta}_\infty(t) = \frac{\beta_\infty t}{1 + \beta_\infty t} \]  
(65)

Note (55) together with (57) imply

\[ \hat{\mu}_\infty(T) = \left( \frac{\hat{\lambda}_{B.m}}{\beta_\infty} \right) \ln \left(1 + \beta_\infty T \right) = m \]  
(66)

This agrees with intuition in the sense that \( \hat{\mu}_\infty(T) \) is an estimate of the expected number of distinct B-modes generated over the test period [0,T] while \( m \) is the observed number of distinct B-modes that occur.

Suppose we adopt the view that our “model of reality” for a system or subsystem is the AMPM for a finite \( K \) which is large but unknown. Then we can consider the limiting AMPM projection estimates as approximations to the AMPM estimates that correspond to the “true” value of \( K \). Our discussion in this section suggests that over the projection range of \( t \geq T \) values of practical interest, the limiting estimates should be good approximations for complex systems or subsystems. In this sense, knowing the “true” value of \( K \) is usually unimportant. Note, however, it is useful to have available the computational formulas for the finite \( K \) AMPM projection estimators as a function of \( K \).

For example, we can compare the graphs of a projection estimator such as \( \hat{\rho}_K(t) \) or \( \hat{\mu}_K(t) \) over the range of interest for different values of \( K \) to the corresponding limiting estimator. In this fashion we can discern the nature of the convergence, for example, the
rapidity of convergence and whether the convergence is strictly increasing or decreasing for \( t \) values of interest. This type of graphical analysis is illustrated with an example.

### 1.4.6 Example

We shall illustrate several key features of our projection model and associated estimators by applying the model to a data set generated during an Army system development program. Here, we shall just focus on the B-modes and let \( \hat{\lambda}_A = 0 \). This test data set consists of \( m = 163 \) B-mode first occurrence times generated over \( T = 8000 \) "equivalent" mission hours.

In Figure 1, we display the cumulative number of distinct B-modes versus the mission hours. We also display the graphs of \( \hat{\mu}_k(t) \) for several values of \( K \). We can show that the greatest lower bound, \( K_0 \), for the set of \( K \)-values for which the AMPM estimators are well defined corresponds to a degenerate gamma. This limiting gamma density has zero variance and mean equal to \( \lambda \), where \( \lambda_i = \lambda \) for \( i = 1, \ldots, K \). To avoid numerical instability, separate maximum likelihood equations were derived and used for this limiting case. On our graphs we have labeled the curves associated with this case (i.e., \( K = K_0 \)) IBM to indicate that this limiting form for \( \mu(t) \) coincides with the IBM model [5]. More explicitly, the IBM model uses this \( \mu(t) \) for the expected number of "non-random" failures experienced in \( t \) test hours. This limiting form for \( \mu(t) \) also is used by Musa in his software reliability basic execution time model [6]. It is interesting to note that the opposite AMPM limiting form, \( \mu_\alpha(t) \), is used by Musa and Okumoto in their Logarithmic Poisson software reliability execution time model [7]. In both of Musa's models, \( \mu(t) \) represents the expected number of software failures experienced over test period \([0, t]\), where \( t \) denotes execution time.

Note over the data range, i.e., \( 0 \leq t \leq 8000 \) hours, the graphs of \( \hat{\mu}_k(t) \) are visually indistinguishable for \( K_{IBM} \leq K \leq \infty \). In such circumstances the value of \( K \) cannot be reasonably assessed from the test data even if one can formally obtain an mle for \( K \). In fact, applying the IBM model (where all \( \lambda_i \) are implicitly assumed to be equal), we can always obtain an mle for \( K \) whenever

\[
\frac{1}{m} \sum_{i=1}^{m} t_i \leq \frac{T}{2}
\]

(see Musa, Iannino, and Okumoto with respect to the exponential class family [8]). However, it has been our experience that the IBM estimate of \( K \), \( K_{IBM} = K_0 \), is often only marginally higher than \( m \), the observed number of distinct B-modes. Since \( \hat{\mu}_k(t) \) approaches \( K \) as \( t \to \infty \), such a low estimate of \( K \) forces the slope of \( \hat{\mu}_k(t) \) to quickly approach zero beyond \( T \) (Figure 2). Note \( \hat{\mu'}_k(t) \) is the slope of \( \hat{\mu}_k(t) \). Thus we can see that such a low estimate of \( K \) quickly forces \( \hat{\mu'}_k(t) \) close to zero for \( t \geq T \). This in turn
tends to produce an "optimistic" failure intensity projection, especially when the assessed value of $\mu_d$ is high. This follows from the formula

$$\tilde{\bar{\rho}}_K(t) = \tilde{\lambda}_A + (1 - \mu_d^* ) \tilde{\lambda}_{A,K} + \mu_d^* \tilde{\lambda}_K(t)$$

which applies for $K_{IBM} \leq K \leq \infty$. Thus a good fit over [0, T] is not a sufficient condition to ensure that a projection model will provide reasonable projection estimates for $t \geq T$.

Looking at Figure 3, as one might expect, the model with $K = \infty$ appears to provide a more conservative estimate of $\bar{\rho}_K(t)$ for $t \geq T$ than do the finite $K$ estimators. However, for $t \geq T$, it is important to note that the $\bar{\mu}_K(t)$, $\bar{\rho}_K(t)$ and $\tilde{\theta}_K(t)$ graphs, displayed in Figures 2, 3, and 4, respectively, quickly become much closer to the corresponding $K = \infty$ graph than to the $K = K_{IBM}$ graph as $K$ increases above $K_{IBM}$.

Observe from Figure 4, $\tilde{\theta}_K(8000) \approx .67$ for $K_{IBM} < K < \infty$. Thus, whatever the "true" value of $K$, we estimate that the remaining B-modes contribute about $.33 \lambda_q$ to the system failure intensity.
Figure 1. Observed Versus Estimate of Expected Number of B-Modes.

Figure 2. Extrapolation of Estimated Expected Number of B-Modes As Function of K.
(Data Ends at 8000 Hours)
Figure 3. Projected MTBF for Different K's.  
(Based on Initial 8000 Hours of Data)

Figure 4. Estimated Fraction of Expected Initial B-Mode Failure
Intensity Surfaced for Different K's.  
(Based on Initial 8000 Hours of Data)
REFERENCES


