3 RELIABILITY GROWTH TRACKING

3.1 Introduction. This section contains material from MIL-HDBK-189 [1] on the AMSAA Continuous Tracking Model. In addition, it presents the AMSAA Discrete Tracking Model developed in [2] and an AMSAA subsystem level tracking model (SSTRACK) from [3].

3.1.1 Definition and Objectives of Reliability Growth Tracking. Reliability growth tracking is a process that allows management the opportunity to gauge the progress of the reliability effort for a system by obtaining a demonstrated numerical measure of the system reliability during a development program based on test data. Some objectives of reliability growth tracking include:

- determining if system reliability is increasing with time (i.e., growth is occurring) and to what degree (i.e., growth rate), and

- estimating the demonstrated reliability - a reliability estimate based on test data for the system configuration under test at the end of the test phase. This latter estimate is based on the actual performance of the system tested and not on some future configuration.

Reliability growth tracking allows for the situation where the configuration of the system may be changing as a result of the incorporation of corrective actions to problem failure modes. In the presence of reliability growth, the data from earlier configurations may not be representative of the current configuration of the system. On the other hand, the most recent test data, which would best represent the current system configuration, may be limited so that an estimate based upon the recent data would not, in itself, be sufficient for a valid determination of reliability. Because of this situation, reliability growth tracking may offer a viable method for combining test data from several configurations to obtain a demonstrated reliability estimate for the current system configuration, provided the reliability growth tracking model adequately represents the combined test data.

3.1.2 Managerial Role. The role of management in the reliability growth tracking process is twofold:

- to systematically plan and assess reliability achievement as a function of time and other program resources (such as personnel, money, available prototypes, etc..) and,

- to control the ongoing rate of reliability achievement by the addition to or reallocation of these program resources based on comparisons between the planned and demonstrated reliability values.

To achieve reliability goals, it is important that the program manager be aware of reliability problems during the conduct of the development program so that effective system design changes can be funded and implemented. It is essential, therefore, that periodic assessments (tracking) of reliability be made during the test program (usually at the end of a test phase) and compared to the planned reliability goals. A comparison between the assessed and planned
values will suggest whether the development program is progressing as planned, better than
planned, or not as well as planned. Thus, tracking the improvement in system reliability through
quantitative assessments of progress is an important management function.

3.1.3 Types of Reliability Growth Tracking Models. Reliability growth tracking
models are distinguished according to the level at which testing is conducted and failure data are
collected. They fall into two categories: system level and subsystem level. For system level
reliability growth tracking models, testing is conducted in a full-up integrated manner, failure
data are collected on an overall system basis, and an assessment is made regarding the system
reliability. For subsystem level reliability growth tracking models, the subsystems are tested and
the failure data are collected on an individual subsystem basis -- the subsystem data are then
"rolled up" to arrive at an estimate for the demonstrated system reliability.

System level reliability growth tracking models are further classified according to the
usage of the system. They fall into two groups -- continuous and discrete models -- and are
defined by the type of outcome that the usage provides. Continuous models are those that apply
to systems for which usage is measured on a continuous scale, such as time in hours or distance
in miles. For continuous models, outcomes are usually measured in terms of an interval or
range; for example, mean time/miles between failures. Discrete models are those that apply to
systems for which usage is measured on an enumerative or classificatory basis, such as pass/fail
or go/no-go. For discrete models, outcomes are recorded in terms of distinct, countable events
that give rise to probability estimates.

3.1.4 Model Substitution.

List of Notation

Discrete Parameters:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
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<tr>
<td>N</td>
<td>number of trials = sample size</td>
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<tr>
<td>S</td>
<td>success</td>
</tr>
<tr>
<td>F</td>
<td>failure</td>
</tr>
<tr>
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<td>number of successes</td>
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<td>number of failures</td>
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<td>unreliability</td>
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<td>reliability</td>
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Continuous Parameters:

<table>
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<td>MTTF</td>
<td>mean time/trials to failure</td>
</tr>
<tr>
<td>MTBF</td>
<td>mean time/trials between failures</td>
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</table>

In general, continuous models are designed for continuous data, and discrete models are
designed for discrete data. In the event a designated model is unavailable for use, it may be
possible to use a continuous model for discrete data or a discrete model for continuous data. The
latter case is generally not a practical option, though. (The AMSAA Subsystem Tracking Model,
tor example, is a continuous model that may be used with discrete data, subject to the conditions mentioned at the end of this paragraph.) In cases involving model substitution, the “substitute” model is used as an approximation for the intended model, and the original data appropriate for the intended model must be converted to a format appropriate for the substitute model. Note that in applying a continuous model to discrete data, the results of the approximation improve as the number of trials increases and the probability of failure decreases.

By way of an example, we show a method for converting discrete data to a continuous format and vice versa. Suppose that from a sample size of N = 5 trials the following outcomes are observed, where S denotes a success and F denotes a failure:

\[
S \quad S \quad S \quad S \quad F
\]

The number of successes, NS, is four; the number of failures, NF, is one; and \( N = NS + NF \).

To begin, note that in discrete terms:

\[
U = \text{probability(failure)} = \frac{NF}{N}
\]

(1)

The reciprocal of \( U \), namely \( N/NF \), may be viewed as a measure of the number of trials to the number of failures, MTTF, thus allowing a continuous measure to be related to a discrete measure:

\[
MTTF = \frac{1}{U}
\]

(2)

In the example, \( MTTF = 5 \) and \( MTBF = 4 \), so that:

\[
MTBF = MTTF - 1
\]

(3)

Substituting (2) into (3) and noting that \( R = 1 - U \) results in:

\[
MTBF = \frac{1}{U} - 1 = \frac{R}{1 - R}
\]

(4)

Equation (4) is used to convert a discrete measure to a continuous measure. To convert a continuous measure to a discrete measure, rearrange (4) and solve for \( R \):

\[
MTBF + 1 = \frac{1}{1 - R}
\]

(5)

\[
R = 1 - \frac{1}{MTBF + 1}
\]

(6)
3.2 System Level Reliability Growth Tracking Models.

3.2.1 Continuous Tracking Models.

3.2.1.1 Background and Basis for the AMSAA Continuous Tracking Model.

List of Notation

\( t_i \) cumulative test time when design modification \( i \) is made

\( K \) final entry in a sequence of test times; point where the last design modification is made

\( \lambda_i \) constant failure rate during \( i \)-th time interval

\( F_i \) number of failures during \( i \)-th time interval

\( \theta_i \) mean value function for \( F_i \)

\( f \) a particular realization of \( F_i \)

\( e \) exponential function

\( t \) cumulative test time

\( F(t) \) total number of system failures by time \( t \)

\( \theta(t) \) mean value function for \( F(t) \)

\( \rho(y) \) failure rate for configuration \( i \) where \( y \in [t_{i-1}, t_i) \)

\( \rho(t) \) instantaneous system failure rate at time \( t \); also referred to as the failure intensity function

\( \lambda \) scale parameter of parametric function \( \rho(t) \); \( \lambda > 0 \)

\( \beta \) shape parameter of parametric function \( \rho(t) \); \( \beta > 0 \)

\( m(t) \) instantaneous mean time between failures at time \( t \)

\( T \) total test time

\( F \) total observed number of failures by time \( T \)

\( X_i \) cumulative time to \( i \)-th failure

\( \wedge \) denotes an estimate when placed over a parameter

\( L \) lower confidence coefficient

\( U \) upper confidence coefficient

\( \gamma \) desired confidence level

\( - \) denotes an unbiased estimate when placed over a parameter

\( \alpha \) significance level

The AMSAA Continuous Reliability Growth Tracking Model may be used to track the reliability improvement of a system during a development test phase for which usage is measured on a continuous scale. The model may also be used for tracking the reliability of one-shot (discrete) systems if there are a large number of trials and the system demonstrates high reliability during test.
The model is designed for tracking system reliability within a test phase and not across test phases. Accordingly, the basis of the model is described in the following way. Let the start of a test phase be initialized at time zero, and let \( 0 = t_0 < t_1 < t_2 < \ldots < t_x \) denote the cumulative test times on the system when design modifications are made. Assume the system failure rate is constant between successive \( t_i \)'s, and let \( \lambda_i \) denote the constant failure rate during the \( i \)-th time interval \([t_{i-1}, t_i)\). The time intervals do not have to be equal in length. Based on the constant failure rate assumption, the number of failures \( F_i \) during the \( i \)-th time interval is Poisson distributed with mean \( \theta_i = \lambda_i (t_i - t_{i-1}) \).

That is,

\[
\text{Prob}(F_i = f) = \frac{(\theta_i)^f e^{-\theta_i}}{f!} \quad (f = 0, 1, 2, \ldots) \tag{7}
\]

During developmental testing programs, if more than one system prototype is tested and if the prototypes have the same basic configuration between modifications, then under the constant failure rate assumption, the following are true:

- the time \( t_i \) may be considered as the cumulative test time to the \( i \)-th modification, and

- \( F_i \) may be considered as the cumulative total number of failures experienced by all system prototypes during the \( i \)-th time interval \([t_{i-1}, t_i)\).

The previous discussion is summarized graphically:
Let $t$ denote the cumulative test time, and let $F(t)$ be the total number of system failures by time $t$. If $t$ is in the first time interval:

$$0 \quad t \quad t_1 \quad t_2 \quad t_3 \quad t_4$$

then $F(t)$ has the Poisson distribution with mean $\lambda_1 t$. Now if $t$ is in the second time interval:

$$0 \quad t_1 \quad t \quad t_2 \quad t_3 \quad t_4$$

then $F(t)$ is the number of system failures $F_i$ in the first time interval plus the number of system failures in the second time interval between $t_i$ and $t$. The failure rate for the first time interval is

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Figure 1. Failure Rates Between Modifications.

Figure 2. Time Line for Phase 2 (t in first time interval).

Figure 3. Time Line for Phase 2 (t in second time interval).
and the failure rate for the second time interval is \( \dot{\lambda}_2 \). Therefore, the mean of \( F(t) \) is the sum of the mean of \( F_i = \lambda_i t_i \) plus the mean number of failures from \( t_i \) to \( t \), which is \( \dot{\lambda}_2 (t-t_i) \). That is, \( F(t) \) has mean \( \mathcal{M}(t) = \lambda_1 t_1 + \dot{\lambda}_2 (t-t_i) \).

When the failure rate is constant (homogeneous) over a test interval, then \( F(t) \) is said to follow a homogeneous Poisson process with mean number of failures of the form \( \dot{\lambda} t \). When the failure rates change with time, e.g., from interval 1 to interval 2, then under certain conditions, \( F(t) \) is said to follow a nonhomogeneous Poisson process (NHPP). In the presence of reliability growth, \( F(t) \) would follow a NIIPP with mean value function:

\[
\mathcal{M}(t) = \int_0^t \rho(y) dy
\]

(8)

where \( \rho(y) = \lambda_i, y \in [t_{i-1}, t_i) \). From (7), for any \( t > 0 \),

\[
\text{Prob}[F(t) = f] = \left[ \mathcal{M}(t) \right]^f \frac{e^{-\mathcal{M}(t)}}{f!} \quad (f = 0, 1, 2, \ldots)
\]

(9)

The integer-valued process \( \{F(t), t > 0\} \) may be regarded as a NHPP with intensity function \( \rho(t) \). The physical interpretation of \( \rho(t) \) is that for infinitesimally small \( \Delta t \), \( \rho(t) \Delta t \) is approximately the probability of a system failure in the time interval \( (t, t + \Delta t) \); that is, it is approximately the instantaneous system failure rate. If \( \rho(t) = \dot{\lambda} \), a constant failure rate for all \( t \), then a system is experiencing no growth over time, corresponding to the exponential case. If \( \rho(t) \) is decreasing with time, \( (\dot{\lambda}_1 > \dot{\lambda}_2 > \dot{\lambda}_3 \ldots) \), then a system is experiencing reliability growth. Finally, \( \rho(t) \) increasing over time indicates deterioration in system reliability.

Based on the learning curve approach, which is outlined in detail in the section on the AMSAA Discrete Reliability Growth Tracking Model, the AMSAA Continuous Reliability Growth Tracking Model assumes that \( \rho(t) \) may be approximated by a continuous, parametric function. Using a result established for the Discrete Model:

\[
E[F(t)] = \dot{\lambda} t^\beta
\]

(10)

and the instantaneous system failure rate \( \rho(t) \) is the change per unit time of \( E[F(t)] \):

\[
\rho(t) = \frac{d}{dt} E[F(t)] = \dot{\lambda} \beta t^{\beta - 1} \quad (\dot{\lambda}, \beta, t > 0)
\]

(11)

With a failure rate \( \rho(t) \) that may change with test time, the NHPP provides a basis for describing the reliability growth process within a test phase.
3.2.1.2 The AMSAA Continuous Reliability Growth Tracking Model. The AMSAA Continuous Reliability Growth Tracking Model assumes that within a test phase failures are occurring according to a nonhomogeneous Poisson process with failure rate (intensity of failures) represented by the parametric function:

\[ \lambda(t) = \lambda \beta t^{\beta - 1} \quad (\lambda, \beta, t > 0) \]  \(\text{(12)}\)

where the parameter \(\lambda\) is referred to as the scale parameter because it depends upon the unit of measurement chosen for \(t\), the parameter \(\beta\) is referred to as the growth or shape parameter because it characterizes the shape of the graph of the intensity function (Equation (12) and Figure 4), and \(t\) is the cumulative test time. Under this model the function:

\[ m(t) = \frac{1}{\lambda(t)} = (\lambda \beta t^{\beta - 1})^{-1} \]  \(\text{(13)}\)

is interpreted as the instantaneous mean time between failures (MTBF) of the system at time \(t\). When \(t\) corresponds to the total cumulative time for the system; that is, \(t = T\), then \(m(T)\) is the demonstrated MTBF of the system in its present configuration at the end of test.
Figure 5. Test Phase Reliability Growth Based on AMSAA Continuous Tracking Model.

Note that the theoretical curve is undefined at the origin. Typically the MTBF during the initial test interval \([0, t_1]\) is characterized by a constant reliability with growth occurring beyond \(t_1\).

Cumulative Number of Failures

The total number of failures \(F(t)\) accumulated on all test items in cumulative test time \(t\) is a Poisson random variable, and the probability that exactly \(f\) failures occur between the initiation of testing and the cumulative test time \(t\) is:

\[
\text{Prob}[F(t) = f] = \frac{\vartheta(t)^f e^{-\vartheta(t)}}{f!}
\]

(14)

in which \(\vartheta(t)\) is the mean value function; that is, the expected number of failures expressed as a function of test time. To describe the reliability growth process, the cumulative number of failures is a function of the form \(\vartheta(t) = \lambda t^\beta\), where \(\lambda\) and \(\beta\) are positive parameters.

Number of Failures in an Interval

The number of failures occurring in the interval from test time \(t_1\) until test time \(t_2\), where \(t_2 > t_1\), is a Poisson random variable with mean:

\[
\vartheta(t_2) - \vartheta(t_1) = \lambda \left(t_2^\beta - t_1^\beta\right)
\]

(15)

According to the model assumption, the number of failures that occur in any time interval is statistically independent of the number of failures that occur in any interval which does not overlap the first interval, and only one failure can occur at any instant of time.
Intensity Function
The intensity function in (12) is sometimes referred to as a failure rate; it is not the failure rate of a life distribution, rather it is the failure rate of a process, namely a NHPP.

Option For Individual Failure Time Data

Estimation Procedures For Model
Modeling reliability growth as a nonhomogeneous Poisson process permits an assessment of the demonstrated reliability by statistical procedures. The method of maximum likelihood provides estimates for the scale parameter \( \hat{\lambda} \) and the shape parameter \( \beta \), which are used in the estimation of the intensity function \( \rho(t) \) in (12). In accordance with (13), the reciprocal of the current value of the intensity function is the instantaneous mean time between failures (MTBF) for the system. Procedures for point estimation and interval estimation for the system MTBF are described in more detail. A goodness-of-fit test to determine model suitability is also described.

The procedures outlined in this section are used to analyze data for which (a) the exact times of failure are known and (b) testing is conducted on a time terminated basis or the tests are in progress with data available through some time. The required data consist of the cumulative test time on all systems at the occurrence of each failure as well as the accumulated total test time \( T \). To calculate the cumulative test time of a failure occurrence, it is necessary to sum the test time on every system at the point of failure. The data then consist of the \( F \) successive failure times \( X_1 < X_2 < X_3 < \ldots < X_F \) that occur prior to \( T \). This case is referred to as the Option for Individual Failure Time Data.

Point Estimation
The method of maximum likelihood provides point estimates for the parameters of the failure intensity function (12). The maximum likelihood estimate (mle) for the shape parameter \( \beta \) is:

\[
\hat{\beta} = \frac{F}{F \ln T - \sum_{i=1}^{F} \ln X_i}
\]  

By equating the observed number of failures by time \( T \) (namely \( F \)) with the expected number of failures by time \( T \) (namely \( \text{E}[F(T)] \)) and by substituting mle's in place of the true, but unknown, parameters in (10) we obtain:

\[
F = \hat{\lambda} T^\beta
\]  

from which we obtain an estimate for the scale parameter \( \hat{\lambda} \):

\[
\hat{\lambda} = \frac{F}{T^\beta}
\]  

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For any time $t > 0$, the failure intensity function is estimated by:

$$\hat{\dot{\lambda}}(t) = \hat{\lambda} \hat{\beta} t^{\hat{\beta} - 1}$$  \hspace{1cm} (19)

In particular, (19) holds for the total test time $T$. By substitution from (17), the estimator $\hat{\rho}(T)$ can be written as:

$$\hat{\rho}(T) = \hat{\lambda} \hat{\beta} T^{\hat{\beta} - 1} = \hat{\beta} \left( \frac{\hat{\lambda} T^{\hat{\beta}}}{T} \right) = \hat{\beta} \left( \frac{F}{T} \right)$$  \hspace{1cm} (20)

where $F/T$ is the estimate of the intensity function for a homogeneous Poisson process. Hence the fraction $(1 - \hat{\beta})$ of the initial failure intensity is effectively removed by time $T$, resulting in (20).

Finally, the reciprocal of $\hat{\rho}(T)$ provides an estimate of the mean time between failures of the system at the time $T$ and represents the system reliability growth under the model:

$$\hat{m}(T) = \frac{1}{\hat{\rho}(T)} = \left( \hat{\lambda} \hat{\beta} T^{\hat{\beta} - 1} \right)^{-1}$$  \hspace{1cm} (21)

**Interval Estimation**

Interval estimates provide a measure of the uncertainty regarding a parameter. For the reliability growth process, the parameter of primary interest is the system mean time between failures at the end of test, $m(T)$. The probability distribution of the point estimate for the intensity function at $T$, $\hat{\rho}(T)$, is the basis for the interval estimate for the true (but unknown) value of the intensity function at $T$, $\rho(T)$.

These interval estimates are referred to as confidence intervals and may be computed for selected confidence levels. The values in Table 1 facilitate computation of two-sided confidence intervals for $m(T)$ by providing confidence coefficients $L$ and $U$ corresponding to the lower bound and upper bound, respectively. These coefficients are indexed by the total number of observed failures $F$ and the desired confidence level $\gamma$. The two-sided confidence interval for $m(T)$ is thus:

$$L_{F,\gamma} \hat{m}(T) \leq m(T) \leq U_{F,\gamma} \hat{m}(T)$$  \hspace{1cm} (22)

Table 2 may be used to compute one-sided interval estimates (lower confidence bounds) for $m(T)$ such that:

$$L_{F,\gamma} \hat{m}(T) \leq m(T)$$  \hspace{1cm} (23)
Note that both tables are to be used only for time-terminated growth tests. Also, since the number of failures has a discrete probability distribution, the interval estimates in (22) and (23) are conservative; that is, the actual confidence level is slightly larger than the desired confidence level $\gamma$. 
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<th>L</th>
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\( \gamma \) = confidence level

TABLE 1. LOWER (L) AND UPPER (U) COEFFICIENTS FOR CONFIDENCE INTERVALS FOR MTBF FROM TIME TERMINATED RELIABILITY GROWTH TEST

60
For $F > 100$, \[ I \approx \left(1 + \frac{z_{\gamma/2}}{\sqrt{2F}}\right)^{-2} \]

and \[ II \approx \left(1 - \frac{z_{\gamma/2}}{\sqrt{2F}}\right)^{-2} \]

in which $z_{\gamma/2}$ is the $100\times\left(\frac{5 + \gamma}{2}\right) - th$ percentile of the standard normal distribution.
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Goodness of Fit

For the case where the individual failure times are known, a Cramér-von Mises statistic is used to test the null hypothesis that a nonhomogeneous Poisson process with failure intensity function (12) properly describes the reliability growth of a system. To calculate the statistic, an unbiased estimate of the shape parameter $\beta$ is used:

$$\bar{\beta} = \frac{F - 1}{F} \hat{\beta}$$

(24)

This unbiased estimate of $\beta$ is for a time terminated reliability growth test with $F$ observed failures. The goodness-of-fit statistic is:

$$C_F = \frac{1}{12F} + \sum_{i=1}^{F} \left( \left( \frac{X_i}{T} \right)^\beta - \frac{2i - 1}{2F} \right)^2$$

(25)

where the failure times $X_i$ must be ordered so that $0 < X_1 \leq X_2 \leq \ldots \leq X_F$.

The null hypothesis that the model represents the observed data is rejected if the statistic $C_F$ exceeds the critical value for a chosen significance level $\alpha$. Critical values of $C_F$ for $\alpha = .20, .15, .10, .05, .01$ are shown in Table 3 where the table is indexed by $F$, the total number of observed failures.
TABLE 3. CRAMÈR-VON MISES GOODNESS-OF-FIT TEST FOR INDIVIDUAL FAILURE TIME DATA

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For $F > 100$ use values for $F = 100$.

$\alpha$ = significance level

Besides using statistical methods for assessing model goodness-of-fit, one should also construct an average failure rate plot or a superimposed expected failure rate plot (as shown in Figure 6). These plots, derived from the failure data, provide a graphic description of test results and should always be part of the reliability analysis.

**Example**

The following example demonstrates the option for individual failure time data in which two prototypes of a system are tested concurrently with the incorporation of design changes.

(The data in this example are used subsequently for one of the growth subsystems in the example for the AMSAA Subsystem Tracking Model - SSTRACK.) The first prototype is tested for 132.4 hours, and the second is tested for 167.6 hours for a total of $T = 300$ cumulative test hours. Table 4 shows the time on each prototype and the cumulative test time at each failure occurrence. An asterisk denotes the failed system. There are a total of $F = 27$ failures. Although the occurrence of two failures at exactly 16.5 hours is not possible under the assumption of the
model, such data can result from rounding and are computationally tractable using the statistical estimation procedures described previously for the model. Note that the data are from a time terminated test.

**TABLE 4. TEST DATA FOR INDIVIDUAL FAILURE TIME OPTION**  
(An asterisk denotes the failed system.)

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<td>14</td>
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<td>36.5*</td>
<td>95.5</td>
<td>End</td>
<td>132.4</td>
<td>167.6</td>
<td>300.0</td>
</tr>
</tbody>
</table>

By using the 27 failure times listed under the columns labeled “Cumulative Hours” in Table 4 and by applying (16), (18), (19) and (21) we obtain the following estimates. The point estimate for the shape parameter is \( \hat{\beta} = 0.716 \); the point estimate for the scale parameter is \( \hat{\lambda} = 0.454 \); the estimated failure intensity at the end of the test is \( \hat{\rho}(T) = 0.0645 \) failures per hour; the estimated MTBF at the end of the 300-hour test is \( \hat{m}(T) = 15.5 \) hours. As shown in Figure 6, superimposing a graph of the estimated intensity function (19) atop a plot of the average failure rate (using six 50-hour intervals) reveals a decreasing failure intensity indicative of reliability growth.
Using (22), Table 1 and a confidence level of 90 percent, the two-sided interval estimate for the MTBF at the end of the test is [9.9, 26.1]. These results and the estimated MTBF tracking growth curve (substituting t for T in (21)) are shown in Figure 7.
Finally, to test the model goodness-of-fit, a Cramér-von Mises statistic is compared to the critical value from Table 3 corresponding to a chosen significance level $\alpha = 0.05$ and total observed number of failures $F = 27$. Linear interpolation is used to arrive at the critical value. Since the statistic, 0.091, is less than the critical value, 0.218, we accept the hypothesis that the AMSAA Continuous Reliability Growth Tracking Model is appropriate for this data set.

Option for Grouped Data

List of Notation

- $K$: number of intervals (or groups) or the last group
- $i$: interval number
- $t_i$: time at beginning (or end) of interval
- $F_i$: observed number of failures in interval $[t_{i-1}, t_i)$
- $t_K$: total test time
- $\hat{\cdot}$: denotes an estimate when placed over a parameter
- $\beta$: shape parameter ($\beta > 0$)
\[ \lambda \] scale parameter \((\lambda > 0)\)

\[ \lambda(t) \] instantaneous failure intensity at time \(t\)

\[ m(t) \] instantaneous MTBF at time \(t\)

\[ M_K \] MTBF for the last group

\[ E_K \] expected number of failures in the last group

\[ \rho_K \] failure intensity for the last group

\[ F \] total observed number of failures

\[ L \] lower confidence coefficient

\[ U \] upper confidence coefficient

\[ \gamma \] specified confidence level

\[ E_i \] expected number of failures in interval \(i\)

\[ K_R \] number of intervals after recombination of intervals

\[ O_i \] observed number of failures in interval \(i\)

\[ \chi^2 \] chi-squared value

Reliability growth parameters can be estimated in accordance with the AMSAA Continuous Tracking Model even if the exact times of failure are unknown and all that is known is the number of failures that occurred in each interval of time, provided there are at least three intervals and at least two intervals have failures. This case is referred to as the Option for Grouped Data. This section describes the estimation procedures and goodness-of-fit procedures for analyzing such data and provides an example of model usage. In the following discussion, the words “group” and “interval” are interchangeable.

Estimation Procedures for Model

The required data consist of the total number of failures in each of \(K\) intervals of test time. The first interval always starts at test time zero so that \(t_0 = 0\). The groups do not have to be of equal length. The observed number of failures in the interval from \(t_{i-1}\) to \(t_i\) is denoted by \(F_i\).

Point Estimation

The method of maximum likelihood provides point estimates for the parameters of the model. The maximum likelihood estimate for the shape parameter \(\beta\) is the value that satisfies the following nonlinear equation:

\[
\sum_{i=1}^{K} F_i \left[ \frac{t_i^\beta \ln t_i - t_{i-1}^\beta \ln t_{i-1}}{t_i^\beta - t_{i-1}^\beta} - \ln K \right] = 0 \tag{26}
\]

in which \(t_0 \ln t_0\) is defined as zero.
By equating the total expected number of failures to the total observed number of failures:

\[ \lambda t^\beta K = \sum_{i=1}^{K} t_i \]  

(27)

and solving for \( \lambda \), we obtain an estimate for the scale parameter:

\[ \hat{\lambda} = \frac{\sum_{i=1}^{K} t_i}{t^\beta K} \]  

(28)

Point estimates for the intensity function \( \rho(t) \) and the mean time between failures function \( m(t) \) are calculated as in the previous section that describes the Option for Individual Failure Time Data; that is,

\[ \hat{\rho}(t) = \hat{\lambda} \hat{\beta} t^{\hat{\beta}-1} \quad \left( \hat{\lambda}, \hat{\beta}, t > 0 \right) \]  

(29)

\[ \hat{m}(t) = \left[ \hat{\rho}(t) \right]^{-1} \quad \left( \hat{\lambda}, \hat{\beta}, t > 0 \right) \]  

(30)

The functions in (29) and (30) provide instantaneous estimates that give rise to smooth continuous curves, but these functions do not describe the reliability growth that occurs on a configuration basis representative of grouped data. Under the model option for grouped data, the estimate for the MTBF for the last group, \( \hat{M}_K \), is the amount of test time in the last group divided by the estimated expected number of failures in the last group.

\[ \hat{M}_K = \frac{t_K - t_{K-1}}{\hat{E}_K} \]  

(31)

where the estimated expected number of failures in the last group \( \hat{E}_K \) is:

\[ \hat{E}_K = \hat{\lambda} \left( t^\beta_K - t^\beta_{K-1} \right) \]  

(32)

From (31) we obtain an estimate for the failure intensity for the last group:

\[ \hat{\rho}_K = \frac{1}{\hat{M}_K} \]  

(33)

Interval Estimation

Approximate lower confidence bounds and two-sided confidence intervals may be computed for the MTBF for the last group. Using (31) and Table 1, a two-sided approximate confidence interval for \( M_K \) may be calculated from:

69
\[ L_{F, r} \hat{M}_K \leq M_K \leq U_{F, r} \hat{M}_K \]  

(34)

and using (31) and Table 2, a one-sided approximate interval estimate for \( M_K \) may be calculated from:

\[ L_{F, r} \hat{M}_K \leq M_K \]  

(35)

where \( F \) is the total observed number of failures and \( \gamma \) is the desired confidence level.

Goodness-of-Fit

A chi-squared goodness-of-fit test is used to test the null hypothesis that the AMSAA Continuous Reliability Growth Tracking Model adequately represents a set of grouped data. The expected number of failures in the interval from \( t_{i-1} \) to \( t_i \) is approximated by:

\[ \hat{E}_i = \lambda \left( t_i^\beta - t_{i-1}^\beta \right) \]  

(36)

Adjacent intervals may have to be combined so that the estimated expected number of failures in any combined interval is at least five. Let the number of intervals after this recombination be \( K_a \), and let the observed number of failures in the \( i \)-th new interval be \( O_i \) and the estimated expected number of failures in the \( i \)-th new interval be \( \hat{E}_i \). Then the statistic:

\[ \chi^2 = \sum_{i=1}^{K_a} \frac{(O_i - \hat{E}_i)^2}{\hat{E}_i} \]  

(37)

is approximately distributed as a chi-squared random variable with \( K_a - 2 \) degrees of freedom. The null hypothesis is rejected if the \( \chi^2 \) statistic exceeds the critical value for a chosen significance level. Critical values for this statistic can be found in tables of the chi-squared distribution.

Besides using statistical methods for assessing model goodness-of-fit, one should also construct an average failure rate plot or a superimposed expected failure rate plot (as shown in Figure 6). Derived from the failure data, these plots provide a graphic description of test results and should always be part of the reliability analysis.

Example

The following example uses aircraft data to demonstrate the option for grouped data (the data in this example are used subsequently for one of the growth subsystems in the example for the AMSAA Subsystem Tracking Model - SSTRACK.) In this example, an aircraft has scheduled inspections at intervals of twenty flight hours. For the first 100 hours of flight testing the results are:
TABLE 5. TEST DATA FOR GROUPED DATA OPTION

<table>
<thead>
<tr>
<th>Start Time</th>
<th>End Time</th>
<th>Observed Number of Failures</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20</td>
<td>13</td>
</tr>
<tr>
<td>20</td>
<td>40</td>
<td>16</td>
</tr>
<tr>
<td>40</td>
<td>60</td>
<td>5</td>
</tr>
<tr>
<td>60</td>
<td>80</td>
<td>8</td>
</tr>
<tr>
<td>80</td>
<td>100</td>
<td>7</td>
</tr>
</tbody>
</table>

There are a total of $F = 49$ observed failures from $K = 5$ intervals. Solution of (26) for $\hat{\beta}$ yields an estimate of 0.753 for the shape parameter. From (28) the scale parameter estimate is 1.53. For the last group, the intensity function estimate is 0.379 failures per flight hour and the MTBF estimate is 2.6 flight hours. Table 6 shows that those adjacent intervals do not have to be combined after applying (36) to the original intervals. Therefore, $K_R = 5$.

TABLE 6. OBSERVED VERSUS ESTIMATE OF EXPECTED NUMBER OF FAILURES FOR TEST DATA FOR GROUPED DATA OPTION

<table>
<thead>
<tr>
<th>Start Time</th>
<th>End Time</th>
<th>Observed Number of Failures</th>
<th>Estimated Expected Number of Failures</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20</td>
<td>13</td>
<td>14.59</td>
</tr>
<tr>
<td>20</td>
<td>40</td>
<td>16</td>
<td>9.99</td>
</tr>
<tr>
<td>40</td>
<td>60</td>
<td>5</td>
<td>8.77</td>
</tr>
<tr>
<td>60</td>
<td>80</td>
<td>8</td>
<td>8.07</td>
</tr>
<tr>
<td>80</td>
<td>100</td>
<td>7</td>
<td>7.58</td>
</tr>
</tbody>
</table>

To test the model goodness-of-fit, a chi-squared statistic of 5.5 is compared to the critical value of 7.8 corresponding to 3 degrees of freedom and a 0.05 significance level. Since the statistic is less than the critical value, the applicability of the model is accepted.

3.2.2 AMSAA Discrete Tracking Model.

3.2.2.1 Background and Motivation for Model.

List of Notation

- $t$: cumulative test time
- $K(t)$: cumulative number of failures by time $t$
- $c(t)$: cumulative failure rate by time $t$
- $\ln$: natural logarithm function (base $e$)
- $\delta$: constant term representing the y-intercept of a linear equation
- $\alpha$: constant term representing the slope of a linear equation
- $\lambda$: scale parameter ($\lambda > 0$) of power function
\( \beta \)  
shape parameter \( (\beta > 0) \) of power function. \( \beta = 1 - \alpha \)

\( i \)  
configuration number

\( T_i \)  
cumulative number of trials through configuration \( i \)

\( \Sigma \)  
summation of

\( N_i \)  
number of trials in configuration \( i \)

\( K_i \)  
cumulative number of failures through configuration \( i \)

\( M_i \)  
number of failures in configuration \( i \)

\( E[K_i] \)  
expected value of \( K_i \)

\( f_i \)  
probability of failure for configuration \( i \)

\( g_i \)  
probability of failure for trial \( i \)

\( R_i \)  
reliability for configuration \( i \) (or trial \( i \))

\( ^\wedge \)  
denotes an estimate when placed over a parameter.

Reliability growth tracking methodology may also be applied to discrete data in a manner that is consistent with the learning curve property observed by J.T. Duane for continuous data. Accordingly, this section describes model development and maximum likelihood estimation procedures for assessing system reliability for one-shot systems during development.

The motivation for the AMSAA Discrete Reliability Growth Tracking Model comes from the learning curve approach for continuous data as follows.

Let \( t \) denote the cumulative test time, and let \( K(t) \) denote the cumulative number of failures by time \( t \). The cumulative failure rate, \( c(t) \), is the ratio:

\[
c(t) = \frac{K(t)}{t}
\]

(38)

While plotting test data from generators, hydro-mechanical devices and aircraft jet engines, Duane observed that the logarithm of the cumulative failure rate was linear when plotted against the logarithm of the cumulative test time:

\[
\ln c(t) = \delta - \alpha \ln t
\]

(39)

By letting \( \delta = \ln \lambda \) for the \( y \)-intercept and by exponentiating both sides of (39), the cumulative failure rate becomes:

\[
c(t) = \lambda t^{-\alpha}
\]

(40)

By substitution from (38),

\[
\frac{K(t)}{t} = \lambda t^{-\alpha}
\]

(41)

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Multiplying both sides of (41) by \( t \) and letting \( \beta = 1 - \alpha \), the cumulative number of failures by \( t \) becomes:

\[
K(t) = \lambda t^\beta
\]  

(42)

This power function of \( t \) is the learning curve property for \( K(t) \), where \( \lambda, \beta > 0 \).

### 3.2.2.2 Model Development

To construct the AMSAA Discrete Reliability Growth Tracking Model, we use the power function developed from the learning curve property for \( K(t) \) to derive an equation for the probability of failure on a configuration basis. We refer to this situation where growth takes place on a configuration basis (and the number of trials in at least one of the configurations is greater than one) as the grouped data option. In the presence of reliability growth, the failure probability trend for the grouped data option appears graphically as a sequence of decreasing, horizontal steps.

We then note the special case where the configuration size is one for all configurations, develop an equation for the probability of failure, and refer to this special case as the option for trial by trial data. In a growth situation, the failure probability trend for this option is described graphically as a decreasing, smooth curve.

Model development proceeds as follows. Suppose system development is represented by \( i \) configurations. (This corresponds to \( i - 1 \) configuration changes, unless fixes are applied at the end of the test phase, in which case there would be \( i \) configuration changes.) Let \( N_i \) be the number of trials during configuration \( i \), and let \( M_i \) be the number of failures during configuration \( i \). Then the cumulative number of trials through configuration \( i \), namely \( T_i \), is the sum of the \( N_i \) for all \( i \):

\[
T_i = \sum N_i
\]  

(43)

and the cumulative number of failures through configuration \( i \), namely \( K_i \), is the sum of the \( M_i \) for all \( i \):

\[
K_i = \sum M_i
\]  

(44)

We express the expected value of \( K_i \) as \( E[K_i] \) and define it as the expected number of failures by the end of configuration \( i \). Applying the learning curve property to \( E[K_i] \) implies:

\[
E[K_i] = \lambda T_i^\beta
\]  

(45)

We introduce a term for the probability of failure for configuration one, namely \( f_1 \), and use it to develop a generalized equation for \( f_i \) in terms of the \( T_i \) and \( N_i \). From (45), the expected number of failures by the end of configuration one is:
E[ K_1 ] = \lambda T_1^\alpha = f_1 N_1 \Rightarrow f_1 = \frac{\lambda T_1^\beta}{N_1} \quad (46)

Applying (45) again and noting that the expected number of failures by the end of configuration two is the sum of the expected number of failures in configuration one and the expected number of failures in configuration two, we obtain:

E[K_1] = \lambda T_1^\beta = f_1 N_1 + f_2 N_2 = \lambda T_1^\beta + f_2 N_2 \Rightarrow f_2 = \frac{\lambda T_2^\beta - \lambda T_1^\beta}{N_2} \quad (47)

By this method of inductive reasoning we obtain a generalized equation for the failure probability, f_i, on a configuration basis:

f_i = \frac{\lambda T_i^\beta - \lambda T_{i-1}^\beta}{N_i} \quad (48)

and use (48) for the grouped data option.

For the special case where N_i = 1 for all i, (48) becomes a smooth curve, g_i, that represents the probability of failure for the option for trial by trial data:

g_i = \lambda i^\beta - \lambda (i-1)^\beta \quad (49)

In (49), i represents the trial number. Note that T_0 = 0, so that (48) reduces to (46) when i = 1.
Also, for i = 1 in (49), g_i = \lambda. Using (48) we obtain an equation for the reliability (probability of success) for the i-th configuration:

R_i = 1 - f_i \quad (50)

and using (49) we obtain an equation for the reliability for the i-th trial:

R_i = 1 - g_i \quad (51)

Equations (48), (49), (50) and (51) are the exact model equations for tracking the reliability growth of discrete data using the AMSAA Discrete Reliability Growth Tracking Model.

3.2.2.3 Estimation Procedures. This section describes procedures for estimating the parameters of the AMSAA Discrete Reliability Growth Tracking Model. It also includes an approximation equation for calculating reliability lower confidence bounds and an example illustrating these concepts.

The estimation procedures described below provide maximum likelihood estimates (MLE's) for the model's two parameters, \lambda and \beta, where \lambda is the scale parameter and \beta is the
shape (or growth) parameter. The mle's for \( \lambda \) and \( \beta \) allow for point estimates for the probability of failure:

\[
\hat{f}_i = \frac{\lambda \hat{T}_i^\beta - \hat{T}_{i-1}^\beta}{N_i} - \frac{\lambda \left(T_i^\beta - T_{i-1}^\beta\right)}{N_i}
\] (52)

and the probability of success (reliability):

\[
\hat{R}_i = 1 - \hat{f}_i
\] (53)

for each configuration \( i \).

Point Estimation

Let \( \hat{\lambda} \) and \( \hat{\beta} \) be the mle's for \( \lambda \) and \( \beta \) respectively, i.e. let \((\hat{\lambda}, \hat{\beta}) \) such that \((\lambda, \beta)\) maximizes the discrete model likelihood function over the region \( 0 \leq R_i \leq 1 \) for \( i = 1, \ldots, K \). Let \( \hat{R}_i \) denote the corresponding estimate of \( R_i \). If \( 0 < \hat{R}_i < 1 \) for \( i = 1, \ldots, K \) then the point \((\lambda, \beta) = \left(\hat{\lambda}, \hat{\beta}\right)\) satisfies the following likelihood equations:

\[
\sum_{i=1}^{K} \left[ \hat{\lambda} T_i^\beta \ln T_i - \hat{T}_{i-1}^\beta \ln T_{i-1} \right] \left[ \frac{M_i}{\hat{\lambda} T_i^\beta - \hat{T}_{i-1}^\beta} - \frac{N_i - M_i}{N_i - \hat{\lambda} T_i^\beta + \hat{T}_{i-1}^\beta} \right] = 0
\] (54)

and

\[
\sum_{i=1}^{K} \left[ T_i^\beta - T_{i-1}^\beta \right] \left[ \frac{M_i}{\hat{\lambda} T_i^\beta - \hat{T}_{i-1}^\beta} - \frac{N_i - M_i}{N_i - \hat{\lambda} T_i^\beta + \hat{T}_{i-1}^\beta} \right] = 0
\] (55)

We recommend using the model mle's only for this case. Situations can occur when the likelihood is maximized at a point \((\hat{\lambda}, \hat{\beta})\) such that \( \hat{R}_i = 0 \) and \((\hat{\lambda}, \hat{\beta})\) does not satisfy Equations (54) and (55). One such case occurs for the trial-by-trial model when a failure occurs on the first trial. If one wishes to use the model in such an instance we suggest either (i) initializing the model so that at least the first trial is a success or (ii) using the grouped version and initializing with a group that contains at least one success. This should typically produce maximizing values \( \hat{\lambda}, \hat{\beta} \) that satisfy Equations (54) and (55) with \( 0 < \hat{R}_i < 1 \) for \( i = 1, \ldots, K \). Procedure (i) is especially appropriate if performance problems associated with an early design cause the initial failure(s). Since the assessment of the achieved reliability will depend on the model initialization and groupings, the basis for the utilized data and groupings should be considered part of the assessment. A goodness of fit test (such as the chi-squared test discussed in Section 3.2.2.4) should be used to explore whether the model provides a reasonable fit to the data and groupings. If there is insufficient failure data to perform such a test, a binomial point estimate and lower confidence bound based on the total number of successes and trials would provide a
conservative assessment of the achieved reliability $R_K$ under the assumption that $R_K > R_i$ for $i = 1, \ldots, K$.

From (54) and (55) we note the following data requirements for using the model:

Data Requirements

- $K$: number of configurations (or the final configuration)
- $M_i$: number of observed failures for configuration $i$
- $N_i$: number of trials for configuration $i$
- $T_i$: cumulative number of trials through configuration $i$

Interval Estimation

A one-sided interval estimate (lower confidence bound) for the reliability of the final (last) configuration may be obtained from the approximation equation:

$$ LCB_r \approx 1 - \left(1 - \hat{R}_K\right) \left(\frac{\chi^2_{\gamma, n+2}}{n}\right) $$

(56)

where

- $LCB_r$: an approximate lower confidence bound at the gamma ($\gamma$) confidence level for the reliability of the last configuration, where $\gamma$ is a decimal number in the interval $(0,1)$
- $\hat{R}_K$: a maximum likelihood estimate for the reliability of the last configuration
- $n$: the total number of observed failures (summed) over all configurations $i$, ($i = 1..K$)
- $\chi^2_{\gamma, n+2}$: the gamma percentile point of the chi-squared distribution with $n+2$ degrees of freedom

3.2.2.4 Goodness-of-Fit. Provided there is sufficient data to obtain at least five expected number of failures per group, a chi-squared goodness-of-fit test may be used to test the null hypothesis that the AMSAA Discrete Reliability Growth Tracking Model adequately represents a set of grouped discrete data or a set of trial by trial data. If these conditions are met, then one may use the chi-squared goodness-of-fit procedures outlined previously for the Continuous Reliability Growth Tracking Model.

Besides using statistical methods for assessing model goodness-of-fit, one should also construct an average failure rate plot or a superimposed expected failure rate plot (as shown in Figure 6). Derived from the failure data, these plots provide a graphic description of test results and should always be part of the reliability analysis.
3.2.2.5 Example. The following example is an application of the grouped data option of the AMSAA Discrete Reliability Growth Tracking Model for a system having four configurations of development test data:

**TABLE 7. TEST DATA FOR GROUPED DATA OPTION**

<table>
<thead>
<tr>
<th>Configuration Number, i \ K = 4</th>
<th>Observed Number of Failures in Configuration i \ $M_i$</th>
<th>Number of Trials in Configuration i \ $N_i$</th>
<th>Cumulative Number of Trials Through Configuration i \ $T_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>19</td>
<td>33</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>15</td>
<td>48</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>20</td>
<td>68</td>
</tr>
</tbody>
</table>

This is represented graphically as:

\[
(M_1 = 5) \qquad (M_2 = 3) \qquad (M_3 = 4) \qquad (M_4 = 4)
\]

\[
T_1 \quad T_2 \quad T_3 \quad T_4
\]

0 \quad 14 \quad 33 \quad 48 \quad 68

\[
(N_1 = 14) \quad (N_2 = 19) \quad (N_3 = 15) \quad (N_4 = 20)
\]

**Figure 8. Test Data for Grouped Data Option.**

The solution of (54) and (55) provides mle’s for $\lambda$ and $\beta$ corresponding to 0.595 and 0.780, respectively. Using (52) and (53) results in the following table:

**TABLE 8. ESTIMATED FAILURE RATE AND ESTIMATED RELIABILITY BY CONFIGURATION**

<table>
<thead>
<tr>
<th>Configuration Number, i \ K = 4</th>
<th>Estimated Probability for Configuration i \ $\hat{f}_i$</th>
<th>Estimated Reliability for Configuration i \ $\hat{R}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.333</td>
<td>.667</td>
</tr>
<tr>
<td>2</td>
<td>.234</td>
<td>.766</td>
</tr>
<tr>
<td>3</td>
<td>.206</td>
<td>.794</td>
</tr>
<tr>
<td>4</td>
<td>.190</td>
<td>.810</td>
</tr>
</tbody>
</table>
A plot of the estimated failure rate by configuration is:

Figure 9. Estimated Failure Rate by Configuration.
and a plot of the estimated reliability by configuration is:

![Graph showing reliability vs. cumulative number of trials]

**Figure 10. Estimated Reliability by Configuration.**

Finally, (56) is used to generate the following table of approximate LCB’s for the reliability of the last configuration:

**TABLE 9. TABLE OF APPROXIMATE LOWER CONFIDENCE BOUNDS (LCB’S) FOR FINAL CONFIGURATION**

<table>
<thead>
<tr>
<th>Confidence Level</th>
<th>LCB</th>
</tr>
</thead>
<tbody>
<tr>
<td>.50</td>
<td>.806</td>
</tr>
<tr>
<td>.75</td>
<td>.783</td>
</tr>
<tr>
<td>.80</td>
<td>.777</td>
</tr>
<tr>
<td>.90</td>
<td>.761</td>
</tr>
<tr>
<td>.95</td>
<td>.747</td>
</tr>
</tbody>
</table>

3.3 Subsystem Level Reliability Growth Tracking Models.

3.3.1 AMSAA SSTRACK Model Description and Conditions For Usage. The AMSAA Subsystem Tracking Model (SSTRACK) is a tool for assessing system level reliability from lower level test results. The methodology was developed to make greater use of component or subsystem test data in estimating system reliability. By representing the system as a series of...
independent subsystems, the methodology permits an assessment of the system level demonstrated reliability at a given confidence level from the subsystem test data. This system level assessment is permissible provided the:

- subsystem test conditions/usage are in conformance with the proposed system level operational environment (as embodied in the Operational Mode Summary/Mission Profile [OMS/MP]) and

- Failure Definitions/Scoring Criteria (FD/SC) formulated for each subsystem are consistent with the FD/SC used for system level test evaluation.

The SSTRACK methodology supports a mix of test data from growth and non-growth subsystems. Statistical goodness-of-fit procedures are used for assessing model applicability for growth subsystem test data. For non-growth subsystems, the model uses fixed configuration test data in the form of the total test time and the total number of failures. The model applies the Lindström-Madden method [4] for combining the test data from the individual subsystems. Twenty-five subsystems can be represented by the current implementation of the model. SSTRACK is a continuous model, but it may be used with discrete data if the number of trials is large and the probability of failure is small.

A potential benefit of this methodology is that it may allow for reduced system level testing by combining lower level subsystem test results in such a manner that system reliability may be demonstrated with confidence. Another potential benefit is that it may allow for an assessment of the degree of subsystem test contribution toward demonstrating a system reliability requirement. Finally, as mentioned, it may serve as an effective means of combining test data from dissimilar sources, namely growth and non-growth subsystems.

Besides the two provisos stated in the opening paragraph regarding OMS/MP conformance and FD/SC consistency, a caveat in using the methodology is that high-risk subsystem interfaces should be identified and addressed through joint subsystem testing. Also, as in any reliability growth test program, growth subsystem configuration changes must be properly documented for the methodology to provide meaningful results.

The primary output from the SSTRACK computer implementation is a table of approximate lower confidence bounds for the system reliability (MTBF) for a range of confidence levels.

### 3.3.2 Methodology.

**LIST OF NOTATION**

- $\wedge$ denotes an estimate when placed over a parameter
- M Mean Time Between Failures (MTBF)
- D demonstration
- G growth
- LCB Lower Confidence Bound
\[ \gamma \quad \text{gamma = confidence level} \]
\[ T \quad \text{(total) test time} \]
\[ N \quad \text{(total) number of failures} \]
\[ \chi^2_{\alpha, r} \quad \text{chi-squared percentile point for} \, df \, \text{degrees of freedom and} \, \gamma \]
\[ \beta \quad \text{beta = growth parameter from reliability growth tracking model} \]

To be able to handle a mix of test data from growth and non-growth subsystems, the methodology converts all growth subsystem test data to its "equivalent" amount of demonstration test time and "equivalent" number of demonstration failures so that all subsystem results are expressed in a common format; namely, in terms of fixed configuration (non-growth) test data. By treating growth subsystem test data in this way, a standard lower confidence bound formula for fixed configuration test data may be used to compute an approximate system reliability lower confidence bound for the combination of growth and non-growth data. The net effect of this conversion process is that it reduces all growth subsystem test data to "equivalent" demonstration test data while preserving the following two important equivalency properties:

The "equivalent" demonstration data estimators and the growth data estimators must yield:

1. the same subsystem MTBF point estimate and
2. the same subsystem MTBF lower confidence bound.

In other words, the methodology maintains the following relationships, respectively:

\[ \hat{M}_D = \hat{M}_G \quad (57) \]
\[ LCB_r(D) = LCB_r(G) \quad (58) \]

where

\[ \hat{M}_D = \frac{T_D}{N_D} \quad (59) \]
\[ LCB_r(D) = \frac{2T_D}{\chi^2_{2N_D + \gamma, r}} \quad (60) \]

Reducing growth subsystem test data to "equivalent" demonstration test data using the following equations closely satisfies the relationships cited above:

\[ N_D = \frac{N_G}{2} \quad (61) \]
\[ T_D = \frac{\hat{M}_G \times \frac{N_G}{2}}{2\hat{\beta}} \] (62)

The growth estimate for the MTBF, \( \hat{M}_G \), and the estimate for the growth parameter, \( \hat{\beta} \), are described in the sections on point estimation for system level Continuous Reliability Growth Tracking Models.

The model then uses the above equations to compute an approximate lower confidence bound for the serial system reliability (MTBF) from non-growth subsystem demonstration data and growth subsystem “equivalent” demonstration data as described in the following section on the Lindström-Madden method.

3.3.3 Lindström-Madden Method. In addition to using the notation defined in the previous section on Methodology, subsequent equations use the following notation:

**LIST OF NOTATION**

- sys: system level
- min: minimum of
- \( K \): number of subsystems in serial system
- \( \rho \): failure rate
- \( i \): subscript for subsystem number
- \( \Sigma \): summation of

To compute an approximate lower confidence bound (LCB) for the system MTBF from subsystem demonstration and “equivalent” demonstration data, the AMSAA SSTRACK model uses an adaptation of the Lindström-Madden method by computing the following four estimates:

1. the equivalent amount of system level demonstration test time. (This estimate is a reflection of the least tested subsystem because it is the minimum demonstration test time of all the subsystems.),

2. the current system failure rate, which is the sum of the estimated failure rate from each subsystem \( i, i = 1..K \),

3. the “equivalent” number of system level demonstration failures, which is the product of the previous two estimates, and

4. the approximate LCB for the system MTBF at a given confidence level, which is a function of the equivalent amount of system level demonstration test time and the equivalent number of system level demonstration failures.
In equation form, these system level estimates are, respectively:

\[ T_{D,sys} = \min T_{D,i} \quad \text{for } i = 1..K \quad (63) \]

\[ \dot{\rho}_{sys} = \sum_{i=1}^{K} \dot{\rho}_i \quad (64) \]

where

\[ \dot{\rho}_i = \frac{1}{\dot{M}_{D,i}} \quad (65) \]

\[ \dot{M}_{D,i} = \text{the current MTBF estimate for subsystem } i \]

\[ N_{D,sys} = \dot{\rho}_{sys} \times T_{D,sys} \quad (66) \]

\[ LCB_i = \frac{2T_{D,sys}}{\chi_{1,\alpha+2}^2} \quad (67) \]

3.3.4 Example. The following example is an application of the AMSAA Subsystem Level Reliability Growth Tracking Model to a system composed of three subsystems: one non-growth and two growth subsystems. Besides showing that S3TRACK can be used for test data gathered from dissimilar sources (namely, non-growth and growth subsystems), this particular example was chosen to show that system level reliability estimates are influenced by -

- the least tested subsystem and
- the least reliable subsystem, that is, the subsystem with the largest failure rate.

Subsystem 1 in this example is a non-growth subsystem consisting of fixed configuration data of 8,000 hours of test time and 2 observed failures.

Subsystem 2 is a growth subsystem with individual failure time data. In 900 hours of test time there were 27 observed failures occurring at the following cumulative times: 7.8, 49.5, 49.5, 51.0, 64.2, 87.3, 99.9, 169.5, 189.3, 211.8, 219.0, 233.1, 281.7, 286.5, 294.3, 303.3, 396.0, 420.6, 443.1, 447.0, 501.6, 572.1, 579.0, 596.1, 755.7, 847.5, 858.3.

Subsystem 3 is also a growth subsystem with individual failure time data. In 400 hours of test time there were 16 observed failures occurring at the following cumulative times: 15.04, 25.26, 47.46, 53.96, 56.42, 99.57, 100.31, 111.99, 125.48, 133.43, 192.66, 249.15, 285.01, 379.43, 388.97, 395.25.
The following table shows the pertinent statistics for each subsystem i. It is here that all growth (G) subsystem test data are reduced to equivalent demonstration (D) test data.

**TABLE 10. SUBSYSTEM STATISTICS**

<table>
<thead>
<tr>
<th>Statistics (i = 1,2,3)</th>
<th>Subsystem 1 (Non-growth)</th>
<th>Subsystem 2 (Growth)</th>
<th>Subsystem 3 (Growth)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{d,i}$</td>
<td>N/A</td>
<td>900</td>
<td>400</td>
</tr>
<tr>
<td>$N_{G,i}$</td>
<td>N/A</td>
<td>27</td>
<td>16</td>
</tr>
<tr>
<td>$\dot{M}_{u,i}$</td>
<td>N/A</td>
<td>46.53</td>
<td>31.37</td>
</tr>
<tr>
<td>$N_{D,i} = \frac{N_{G,i}}{2}$</td>
<td>2</td>
<td>13.5</td>
<td>8</td>
</tr>
<tr>
<td>$T_{D,i} = \dot{M}<em>{G,i} \times N</em>{D,i}$</td>
<td>8000</td>
<td>628.19</td>
<td>250.95</td>
</tr>
<tr>
<td>$\dot{M}<em>{D,i} = \dot{M}</em>{G,i} \times \frac{T_{D,i}}{N_{D,i}}$</td>
<td>4000</td>
<td>46.53</td>
<td>31.37</td>
</tr>
<tr>
<td>$\hat{\rho}<em>i = \frac{1}{\dot{M}</em>{D,i}}$</td>
<td>$2.50 \times 10^{-4}$</td>
<td>$2.149 \times 10^{-2}$</td>
<td>$3.188 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

System level statistics are computed by applying the Lindström-Madden method to the equivalent demonstration data from each subsystem.

$$T_{D,sys} = \min T_{D,i(i=1,2,3)} = 251.0$$ (68)

$$\hat{\rho}_{sys} = \sum_{i=1}^{3} \hat{\rho}_i = 5.362 \times 10^{-2}$$ (69)

$$\dot{M}_{D,sys} = \frac{1}{\hat{\rho}_{sys}} = 18.7$$ (70)

$$N_{D,sys} = T_{D,sys} \times \hat{\rho}_{sys} = 13.5$$ (71)

$$LCB_{80} = \frac{(2 \times T_{D,sys})}{\chi^2_{2N_{D,sys} + 2, 0.80}} = 14.32 \quad (confidence \ level = 80\%)$$ (72)

Finally, a table of approximate lower confidence bounds is shown for the system reliability (MTBF) for a range of confidence levels.
<table>
<thead>
<tr>
<th>Confidence Level (in percent)</th>
<th>LCB for System MTBF</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>17.77</td>
</tr>
<tr>
<td>55</td>
<td>17.19</td>
</tr>
<tr>
<td>60</td>
<td>16.62</td>
</tr>
<tr>
<td>65</td>
<td>16.07</td>
</tr>
<tr>
<td>70</td>
<td>15.51</td>
</tr>
<tr>
<td>75</td>
<td>14.93</td>
</tr>
<tr>
<td>80</td>
<td>14.32</td>
</tr>
<tr>
<td>85</td>
<td>13.66</td>
</tr>
<tr>
<td>90</td>
<td>12.87</td>
</tr>
<tr>
<td>95</td>
<td>11.82</td>
</tr>
<tr>
<td>98</td>
<td>10.78</td>
</tr>
<tr>
<td>99</td>
<td>10.15</td>
</tr>
</tbody>
</table>
REFERENCES


2. Crow, Larry, Methodology Office Note 1-83, AMSAA Discrete Reliability Growth Model, March 1983

3. Broemm, William, Briefing Charts, Subsystem Tracking Model (SSTRACK) (Demonstration of System Reliability from Lower Level Testing), March 1996